VOL. XX. No. 1. JANUARY, 1910.

THE MONIST

A Quarterly Magazine

Devoted to the Philosophy of Science.

Editor: DR. PAUL CARUS. *Associates:* E. C. HEGELER. MARY CARUS.

CONTENTS:

PAGE

TRUTH (A Poem).
EDITOR 1

THE PERSONALITY OF JESUS IN THE TALMUD.
BERNHARD PICK 4

THE NATURE OF LOGICAL AND MATHEMATICAL THOUGHT.
EDITOR 33

THE FUTURE OF MATHEMATICS.
H. POINCARE 76

TRANSFINITE NUMBERS AND THE PRINCIPLES OF MATHEMATICS (To be continued).
PHILIP E. B. JOURDAIN 93

CRITICISMS AND DISCUSSIONS.
Oddly-Even Magic Squares. D. F. SAVAGE 119
Notes on Oddly-Even Magic Squares. W. S. ANDREWS 126
Bolyai, Lobatchevsky, Russell. GEORGE BRUCE HALSTED 130
A Remark on F. C. Russell's Theorem. YOSHIO MIKAMI 133
A Mathematical Paradox. PHILIP E. B. JOURDAIN 134
On the Problem and Method of Psychology of Religion. L. M. BILLIA 135
The Pragmatist View of Truth. EDITOR 139
William Porcher Du Bose, A Christian Philosopher. EDITOR 144
Psychology More Than a Science. L. M. BILLIA 153
Non-Aristotelian Logic 158
Book Reviews and Notes 160

CHICAGO

THE OPEN COURT PUBLISHING CO.

Price, 60 cents; Yearly, $2.00

LONDON: Kegan Paul, Trench, Trübner & Co., Ltd.

In England and U. P. U., half a crown; Yearly, 9s 6d.

Entered as Second-Class Matter Oct. 10, 1890, at the Post Office at Chicago, Ill., under Act of March 3, 1879.
Copyright by The Open Court Publishing Company, 1909.

THE SCOPE AND CONTENT
OF THE
SCIENCE OF ANTHROPOLOGY

Historical review, library classification and select, annotated bibliography, with a list of the chief publications of leading anthropological societies and museums. By JUUL DIESERUD, A. M.

A thorough-going, painstaking work, in which the author makes an attempt to settle the perplexing question, what anthropology really is, to define its boundaries with other sciences and give a detailed enumeration of its contents as conceived today by the leading anthropologists of the various civilized countries of the world.

The bibliography (95 pages) gives succinct extracts from the works listed, thus furnishing the material on which the historical review and the elaborate library classification are founded.

A most welcome reference book for the anthropologist, and a necessary tool for any library containing anthropological literature.

200 pages, cloth, gilt top, $2.00 net, (8s. 6d.) net.
Also supplied in sheets, $1.50.

The Open Court Publishing Co.

P. O. Drawer F. 378-388 Wabash Ave., Chicago, Ill.

Special Subscription Offer

THE MONIST, for one year	$2.00
THE OPEN COURT, for one year	1.00
Copy of *Philosophy as a Science*	.50
	$3.50
Special Price for All	**$2.50**

THE OPEN COURT PUBLISHING COMPANY
378-388 Wabash Avenue, Chicago

Dear Sirs:

Please find enclosed $2.50, for which please send me THE MONIST and THE OPEN COURT for one year, including complimentary copy of *Philosophy as a Science.*

Yours very truly,

.....

Date..... .....

UMI

UMI

VOL

TRUTH

THE P

THE N

THE F

TRANS

CRITICI

UMI

VOL. XX. No. 1. JANUARY, 1910.

THE MONIST

A Quarterly Magazine

Devoted to the Philosophy of Science.

Editor: DR. PAUL CARUS. *Associates:* E. C. HEGELER. MARY CARUS.

CONTENTS:

PAGE

TRUTH (A Poem).
EDITOR 1

THE PERSONALITY OF JESUS IN THE TALMUD.
BERNHARD PICK 4

THE NATURE OF LOGICAL AND MATHEMATICAL THOUGHT.
EDITOR 33

THE FUTURE OF MATHEMATICS.
H. POINCARE 76

TRANSFINITE NUMBERS AND THE PRINCIPLES OF MATHEMATICS (To be continued).
PHILIP E. B. JOURDAIN 93

CRITICISMS AND DISCUSSIONS.

Oddly-Even Magic Squares. D. F. SAVAGE 119
Notes on Oddly-Even Magic Squares. W. S. ANDREWS 126
Bolyai, Lobatchevsky, Russell. GEORGE BRUCE HALSTED 130
A Remark on F. C. Russell's Theorem. YOSHIO MIKAMI 133
A Mathematical Paradox. PHILIP E. B. JOURDAIN 134
On the Problem and Method of Psychology of Religion. L. M. BILLIA 135
The Pragmatist View of Truth. EDITOR 139
William Porcher Du Bose, A Christian Philosopher. EDITOR 144
Psychology More Than a Science. L. M. BILLIA 153
Non-Aristotelian Logic 158
Book Reviews and Notes 160

CHICAGO
THE OPEN COURT PUBLISHING COMPANY
1910

UMI

COPYRIGHT BY
THE OPEN COURT PUB. CO.
1909

VOL. XX. JANUARY, 1910. No. 1.

THE MONIST

TRUTH.

Πάντων μέτρον ἄνθρωπος·
ἀλλ' ἀνθρώπου μέτρον τὸ Ἓν καὶ Πᾶν.

Truly, the measure of all things is Man;
But Man is measured by the One and All.

Man is a microcosm, and he grows
Unto the stature of full manhood, only
When to the One and All his soul responds.
There is a gauge that measures man, a norm
By which the truth that's in him must be tested.
T'is the Eternal in the change of time,
It is the Law, the Uniformity,
It is the One in this great All,—t'is God!

Mind you, t'is Man, not men, that measures things;
Not I, nor you, nor any other being;
Man only, Man alone.
And what is Man?
Man is the type of mankind,—men's ideal;
Yea, men's ideal!—
Are ideals sham?
Deem you that only things concrete are true?
O, learn to prize the power of ideals
Which more efficient is than Nature's forces
And stronger far than footpound-energy.

Ideals are the factors of man's life;
They are no vain illusions, they are real,
Nay, superreal. Yea, they are Man's guides,—
And they, like guardian angels, help him find
The pre-determined goal of cosmic life.

Man, the ideal, is no transient thing:
He is the cosmic law assuming flesh,
The norm of being in a creature's garb,
An incarnation of the Deity,
Of that All-One which shapes and moulds the world,
Which manifests itself in motes and stars,
And thrills through all their uniformities.
T'is Man, not men, in whom the glory dwells
Of the great One in All,—the Man of Truth.

"Truth changes," sayest thou, and thou art right,
And man himself is changing with his truth.
Both change! for nothing is at rest
In this corporeal world of flux. And yet
Beware lest half-truths lead thy mind astray.

Truth changes as our knowledge broader grows,
As science gains in depth and definition;
But verily the new and broader Truth
Will never call the older Truth a lie,
For lo! it is the selfsame older Truth
As from a higher standpoint it appears,
And all the truths are after all but one.

Truth is no idol, nor an errant light,
T'is not an *ignis fatuus,* no comet.
Truth is a guiding star, a bright ideal,
And like the sailor's compass on high seas,
It leads us gently onward, step by step,

With duly firm-prescribed approximations,
On its own path in definite direction.

Truth is beheld by mind, not by the senses,
T'is not a thing which merchants keep in store,
T'is no commodity which we possess.
Truth is a superhuman power, and
From generation unto generation
Truth marches on, unfolding and revealing
The wondrous mysteries of cosmic life.

Truth is too great that ever it be final.
But Truth remains the Truth e'en though its work
Can never be completed, never finished.
Truth is not untrue, it is not a sham,
Because its resources are ne'er exhausted;
Because its duties, constantly renewed,
From day to day enlarge our field; because
One goal attained entails still further tasks,
And so before our raptured vision stretches
The promised land of vistas infinite.

Truth is life's factor and determinant,
And we are workers in Truth's noble cause.
We yearn for Truth, we need its light; and Truth
Enters our Soul; it takes abode in us,
And consecrates our lives to higher service,
Not we own Truth, t'is Truth that owneth us.

Search for the Truth! Truth's problems are not vain.
Love thou the Truth! trust Truth, and live the Truth!
Walk on Truth's path and Truth will guide thee right.

PAUL CARUS.

UMI

THE PERSONALITY OF JESUS IN THE TALMUD.

INTRODUCTION.

JESUS as represented in the Talmud is a subject which must interest the Christian student. For what can be of profounder interest than to learn what the Jews have said concerning Jesus and Christianity. We naturally look to the Jewish historian Josephus, who described and witnessed the downfall of the Jewish commonwealth. But we are disappointed. True that in his "Antiquities" (XVIII, 3, 3) Josephus has reference to Christ, but scholars are now generally agreed[1] that this passage is a later interpolation. Leaving then aside Josephus, we must turn to that encyclopedia of "Jewish wisdom and unwisdom" which is known as the Talmud. We cannot speak here of the origin and contents of this voluminous work, of which a complete translation into any modern language does not yet exist. We must refer the reader to our article "Talmud."[2] But even this work does not add anything to our knowledge, yea, it is rather disappointing. For the Talmud as we now have it contains not those Christian or rather anti-Christian passages, which it originally had. Modern Judaism complains of the intolerance of the Church, which from the time of Justinian[3] persecuted and burned the Talmud. But it forgets that the Talmud only reaped what

[1] See however among other defenders of the passage in Josephus, Seitz, *Christus-Zeugnisse aus dem klassischen Altertum*, Cologne, 1906, 9 et. seq.

[2] See McClintock & Strong's *Theol. Enc.*, vol. X (1881) *s. v.* "Talmud."

[3] On February 13, 553 he issued a novella "concerning the Jews."

UMI

it has sowed, and that the Church of Rome only acted in accordance with the Talmud itself. For it was the very Talmud which taught that in case of a fire breaking out on the Sabbath, the Gospels and other works of the *minim* (i. e., Christians) should not be rescued. "By the life of my son," said the Rabbi Tarphon, "should they (i. e., these writings) come into my hand I would burn them together with the names of God which they contained. Were I pursued, I would rather take refuge in a temple of idols than in their (i. e., the Christians') houses. For the latter are wilful traitors, while the heathen sinned in ignorance of the right way; and concerning them the Scripture says: 'Behind the doors, also, and the posts, hast thou set up thy remembrance' " (Is. lvii. 8).[4] This fact should not be forgotten. The anti-Christian character of Jewish writings early attracted the attention of Christians, and Agobard, bishop of Lyons (820-830) in his *De Judaicis Superstitionibus*, and Hrabanus Maurus, archbishop of Mayence, in his *Contra Judaeos*, written about 847 A. D., betray acquaintance with Jewish literature. The first attack upon the Talmud was made in the thirteenth century, when A. D. 1240 a conference was held at Paris betwen Nicolas Donin and some Jewish rabbis. When the question came up as to Jesus in the Talmud, rabbi Jechiel, the most prominent of the Jewish rabbis at that conference, would not admit that the Jesus spoken of in the Talmud was Jesus of Nazareth, but another Jesus, a discovery which was copied by some later writers. But modern Jews acknowledge the failure of this argument, for says Dr. Levin in his prize essay: "We must regard the attempt of rabbi Jechiel to ascertain that there were two by the name of Jesus as unfortunate, original as the idea may be."[5] As the author of

[4] Talmud, *Shabbath*, fol. 116, col. 1. This his animosity against Christianity induced some scholars to maintain that this Rabbi Tarphon is the same Trypho who is the interlocutor in Justin Martyr's *Dialogue*.

[5] "Die Religionsdisputation des Rabbi Jechiel von Paris," published in Graetz's *Monatsschrift*, Breslau, 1869, p. 193.

UMI

this essay was a pupil of the rabbinical seminary at Breslau, he certainly expressed the opinion of his teachers. The result of the conference was that the Talmud in wagon-loads was burned at Paris in 1242.

In our days, such accusations against the Talmud were impossible, because all these offensive passages have been removed—not so much by the hands of the Christian censor, as by the Jews themselves—a fact only very often overlooked by controversialists.—In the Jewish year 5391 (i. e., A. D. 1631) a Jewish synod held at Petrikau, in Poland, issued a circular letter[6] to the effect that all such passages in the Talmud which refer to Jesus etc., should be omitted in the future. This letter explains the absence of those offensive passages from the editions of the Talmud published since the publication of the Amsterdam edition in 1644. But happily or unhappily the Jews themselves have taken care that "the expurgated passages in the Talmud" did not become lost to their coreligionists by publishing them anonymously in pamphlets, of which Professor Strack of Berlin mentions no less than four such editions.[7] These collections, published for the most part in Germany, are of a recent date, and are probably intended for more than a mere literary interest.

In order to give back to the Jews what the censor has taken from them and to show them that Christianity has nothing to fear from these expurgated passages, Professor Dalman of Leipsic, one of the few Christian scholars who are entitled to be heard even in *Talmudicis,* has published in a convenient form all these passages contained in the oldest editions of the Talmud and Midrash. To this collection of the censured passages H. Laible appended an introductory essay, and the whole was published under the

[6] The reader can find this circular letter in my article "Talmud" in McClintock & Strong's *Theol. Enc.,* vol. x, p. 172.

[7] The present writer has also one of these collections, published at Cracow, 1893.

title *Jesus Christus im Talmud,* Berlin, 1891, by the missionary "Institutum Judaicum."

Before we enter into the debating club of the rabbis, we will make a few preliminary remarks which can prepare us for the Talmudic statements.

During the life-time of Jesus his miracles were not denied but were traced back to Beelzebub, the prince of the devils (Mark iii. 22). The scribes would not recognize one who sought not their company, but that of publicans and sinners with whom he ate; who broke the Sabbath and abolished the difference between clean and unclean. That the grave of Christ had been empty, the Jews did not deny, but they thought that the disciples had stolen the body (Matt. xxviii. 15). They freely made use of the invective Beelzebub ("master of the house") for the "master" of the Christians as well as for his servants (Matt. x. 25). After his death, the crucified Messiah, as Paul tells us, became unto them a stumbling block (1 Cor. i. 23). The destruction of Jerusalem had made no impression upon "these villains" and upon "that ungodly generation," as Josephus calls his countrymen (*War,* v, 13, 6). But if the sword of Judaism was perforce sheathed, its tongues and pens were active. The apologetic writings of the earliest centuries, show that the Jews were busy throughout this whole era in circulating calumnies against the Christians. Justin Martyr (died 163 A. D.) complains of the Jewish blasphemies against Christ and the Christians. "The high priests of your nation and your teachers," he says, "have caused that the name of Jesus should be profaned and reviled through the whole world" (*Dialogue with Trypho,* 117). "Nay, ye have added thereto, that Christ taught those impious, unlawful, horrible actions, which ye disseminate as charges above all against those who acknowledge Christ as Teacher and as the Son of God" (*ibid.* 108). "Your teachers exhort you to permit

yourselves no conversation whatever with us" (*ibid.* 112). "The Jews regard us as foes and opponents, and kill, and torture us if they have the power. In the lately-ended Jewish war, Bar Kokh'ba, the instigator of the Jewish revolt, caused Christians alone to be dragged to terrible tortures, whenever they would not deny and revile Jesus Christ" (*Apology*, I, 31). "The Jews hate us, because we say that Christ is already come, and because we point out that He, as had been prophesied, was crucified by them" (*Dial.* 35). "Ye have killed the Just and His prophets before Him. And now ye despise those who hope in Him and in God, the King over all and Creator of all things, who has sent Jesus; ye despise and dishonor them, as much as in you lies, in that in your synagogues ye curse those who believe in Christ. Ye only lack the power, on account of those who hold the reins of government, to treat us with violence. But as often as ye have had this power, ye have also done this" (*ibid.* 16). "In your synagogues ye curse all who have become Christians, and the same is done by the other nations, who give a practical turn to the curse, in that when any one acknowledges himself a Christian, they put him to death" (*ibid.* 96).

From the "True Word" of Celsus, which has been answered by Origen, we already learn some of the mean things which the Jews circulated about Jesus. The Jew whom Celsus introduces charges Jesus with having falsely proclaimed himself to be born of a virgin; afterwards he says that Jesus was born in a poor Jewish village, and that his mother was a poor woman of the country, who supported herself with spinning and needlework; that she was cast off by her betrothed, a carpenter; and that after she was rejected by her husband, she wandered about in disgrace and misery till she secretly gave birth to Jesus. Jesus himself was obliged from poverty and necessity to go down as servant into Egypt, where he learnt some of the

secret sciences which are in high honor among the Egyptians; and he placed such confidence in these sciences that on his return to his native land he gave himself out to be a god (I, 28). The Jew of Celsus also declares that the carpenter who was betrothed to Mary, put the mother of Jesus from him, because she had broken faith with him, in favor of a soldier named Panthera (I, 32).

Tertullian writing somewhere about 197-198 A. D. in his *De Spectaculis*, chap. 30, in which he depicts the glorious spectacle of the second coming says that he will turn to the Jews who raged against the Lord and say unto them: "This is your carpenter's son, your harlot's son; your sabbath-breaker, your Samaritan, your demon-possessed! This is He whom ye bought from Judas; this is He who was struck with reeds and fists, dishonored with spittle, and given a draught of gall and vinegar! This is He whom His disciples have stolen secretly, that it may be said He has risen, or the gardener abstracted that his lettuces might not be damaged by the crowds of visitors!"

Such is already the attitude of Judaism towards Jesus at a time when the Talmud was in a state of formation. But if we wish to become acquainted with the Rabbinical Jesus-tradition we must examine the constituent parts of the Talmud, namely the Mishna, Tosephta (i. e., addition or supplement to the Mishna), the Gemara or commentary on the Mishna, and the Midrashim or homiletic literature, especially the Midrash Kohelet or Midrash on Ecclesiastes. The Talmudic Jesus traditions continued themselves even after the time of the completion of the Talmud. They were further developed and enlarged, and reached their full expression in the Middle Ages. In that period the hatred of Jesus which was never quite dormant, begat a literature, in comparison with which the Talmud must be termed almost innocent. The Toldoth Jeshu literature originated, which is still continued. In the Toldoth Jeshu a detailed

UMI

picture of the life of Jesus was put together, of which the authors of the Talmud had no anticipation. The three consonants *j s* (*shin*) *v*, with which the name Jeshu was written, are here explained as being the first letters of the three words: *j* = *jimmach*, *sh* = *sh'mo*, *v* = *v'zichro*, i. e., "may be blotted out his name and his memory"! The Toldoth Jeshu is nothing but the offspring of low fanaticism, malicious delight in defamation, and vulgar imagination which respectable Jews have always despised.

After these preliminaries we now take up the passages of the Talmud as given by Dalman, and which are claimed to refer to Jesus. For the sake of convenience we arrange them under different heads.

BIRTH AND PARENTAGE OF JESUS.

In the Talmud Shabbath 104*b* we read: "He who cuts upon his flesh." It is a tradition that Rabbi Eliezer said to the wise, Has not Ben Stada brought magic spells from Egypt in a cut which was upon his body? They answered him, He was a fool, and we do not take proofs from fools. [Ben Stada is Ben Pandira. Rab Hisda said, The husband was Stada, the paramour was Pandira. The husband was Paphos ben Jehudah, the mother was Stada. The mother was Miriam the dresser of women's hair, as we say in Pumbeditha, Such a one has been false to her husband.][1]

The above passage occurs in a discussion upon the words in the Mishnah which forbid all kinds of writing to be done on the Sabbath. Several kinds are specified, and among them the making of marks upon the flesh. The words at the beginning of the translation, "he who cuts upon his flesh," are the text, so to speak, of the Mishnah which is discussed in what follows. To illustrate the prac-

[1] The passage in [] which also occurs in Sanhedrin 67*a*, is not found in modern editions. It is supplied from Rabbinowicz Diqduqé Sopherim, on the authority of the Munich and Oxford manuscripts, and the older editions.

tice of making marks on the skin, the compilers of the Gemara introduce a tradition, according to which Rabbi Eliezer asked the question, "Did not Ben Stada bring magical spells from Egypt in a cut which was upon his body?" His argument was that as Ben Stada had done this, the practice might be allowable. The answer was that Ben Stada was a fool, and his case proved nothing. Upon the mention however of Ben Stada, a note is added to explain who that person was, and it is for the sake of this note that the passage is quoted.

The two names Ben Stada and Ben Pandera evidently refer to the same person, and that that person is Jesus is shown clearly by the fact that we sometimes meet with the full name "Jeshu son of Pandera," also Jeshu son of Stada." It seems that the question was argued in the schools which of the two familiar designations (son of Stada, son of Pandera) was the correct one. One of the two appellations appeared to be necessarily false. Which was correct?

The subject treated was that the son of Stada had brought charms with him out of Egypt in an incision in his flesh. Thereupon some one objects: the designation Ben Stada is false; he was the son of Pandera. No, says Rab Hisda (a Babylonian teacher, A. D. 217-309), Stada was the name of the husband (of his mother), Pandira the name of her paramour. To call him either the one or the other is therefore correct. To this however is objected that this cannot be true, because the husband is known to have been called Paphos ben Jehudah. Stada must have been not the father but the mother. But how can that be, because the mother was called Miriam the dresser of women's hair? As rejoinder to this follows the conclusion: Of that we are aware, but she is also called Stada, by her nickname. Insomuch as she had a paramour, she was given the "sobriquet" Stada, which consists of the words *stath da,* i. e., she has gone aside, from her husband. Thus

at least the word is explained in the Babylonian Academy at Pumbeditha.

Various attempts have been made to explain the two names Ben Stada and Ben Pandira (also written Pandera, or Pantira). But none of the suggested explanations solves the problem. We leave the two names as relics of ancient Jewish mockery against Jesus, the clue to whose meaning is now lost.

Mention has also been made of Miriam (of which Mary is the equivalent). She is called *m'gaddla nashaia*, i. e., a women's hairdresser. How came the Talmud to bestow this epithet upon the mother of Jesus, for whom elsewhere it has the characteristic designation of adulteress? That Jesus's mother was named Mary, was known to the Jews; that she had born Jesus out of wedlock, was maintained by them. Then they heard a noted Christian woman of Jesus's time often spoken of, who was named Mary of Magdala. What was more natural for those who had already long ceased to ascertain more particularly at the mouth of Christians the history of Jesus, than by this Mary (of) Magdala simply to understand Jesus's mother, especially since their knowledge was confined to one Mary? She was reported to be a great sinner. This harmonized in a twofold way with their assumption, for, that Jesus's mother was a sinner, was maintained by them with the utmost certainty, and now they obtained, as they supposed, actual confirmation of this from the Christians. Miriam (of) Magdala was accordingly the mother of Jesus, and by a name-play the Magdala was turned into *m'gaddla nashaia*, i. e., women's hairdresser.

In the Talmudic passage quoted above we are told that Stada's (i. e., Mary's) lawful husband was Paphos ben Jehudah. Now of this Paphos, who lived a century after Jesus, the Talmud Gittin 90[a] narrates the following:

"There is a tradition, Rabbi Meir used to say: 'Just as

there are various kinds of taste as regards eating, so there are also various dispositions as regards women. There is a man into whose cup a fly falls and he casts it out, but all the same he does not drink it [the cup]. Such was the manner of Paphos ben Jehudah, who used to lock the door upon his wife, and go out.'"

All we learn from this passage directly with regard to Paphos ben Jehudah, a contemporary of Rabbi Akiba, is that he locked up his wife; we are, however, led to conclude, indirectly, that she ultimately proved unfaithful to her tyrannical spouse. What, then, was more simple than for a story-teller to connect this with the details of unfaithfulness found in his Jeshu repertoire? The erring wife was just like Miriam; before long she actually became Miriam, and finally Paphos ben Jehudah was confidently given as Miriam's husband! So they had it in later times, and the great Talmudic commentator Rashi (died A. D. 1105) comments thus upon our passage: "Paphos ben Jehudah was the husband of Mary, the women's hairdresser. Whenever he went out of his house into the street, he locked the door upon her, that no one might be able to speak with her. And that is a course which became him not; for on this account there arose enmity between them, and she in wantonness broke her faith with her husband."

A MARY-LEGEND.

In Talmud Hagigah 4^b we read the following: "When Rab Joseph came to this verse (Prov. xiii. 23), 'But there is that is destroyed without judgment,' he wept. He said: 'Is there really some one who is going [away], when it is not his time?' None but this [told] of Rab Bibi bar Abbai. The Angel of Death was with him. The Angel said to his messenger, 'Go, bring me Miriam the dresser of women's hair.' He brought him Miriam the teacher of chil-

dren. He [the Angel] said, 'I told thee Miriam the dresser of women's hair.' He said, 'If so, I will take this one back.' He said, 'Since thou has brought this one, let her be among the number [of the dead].' "

In this narrative we have a monstrous anachronism. Rab Joseph, who is mentioned here, was born at Shiti, in Babylonia, A. D. 259 and died in 325. Rab Bibi flourished in the fourth century. The latter can neither have seen Mary nor have been her contemporary. The Talmudic commentary Tosaphoth remarks: "The Angel of Death was with him: he related what had already happened, for this about Miriam the dresser of women's hair took place in [the time of] the second temple, for she was the mother of that *so and so* [i. e., Jesus], as it is said in Shabbath 104^b." But the wording of the Talmud says quite distinctly that Mary lived in the very time of Rab Bibi, on which account the Angel of Death spoke with him not of one who had existed earlier, but of one actually living. Further this angel, we may note, at that very time in the presence of Rab Bibi commissions his messenger, to bring her, i. e., deliver her to death. The Tosaphoth notes on Shabbath 104^b seek needlessly to remove the anachronism by the assumption that there were two women's hairdressers, named Mary. But this attempt is in vain, for nothing is known of that second Mary. Besides we must not forget that the Talmud, in relation to Jesus, has no conception of chronology, and indeed, the later the origin of notices about Jesus, the more reckless are they in their chronological lapses. The post-Talmudic Second Targum on the Book of Esther actually reckons Jesus among the ancestors of Haman, an anachronism, which Levy in his Targumic dictionary (I, 330) seeks in vain to justify. In the face of such an unfathomable error what signifies the erroneous representation that Rab Bibi lived in the time of Mary?

JESUS ALLEGED TO BE BORN OUT OF WEDLOCK.

1. *The Pretended Record.*

It is said in Mishnah Jebamoth iv, 13[2] (Gemara 49[b]): "Simeon ben Azai said, 'I have found in Jerusalem a book of genealogies, and therein is written: That *so and so*[3] is a *mamzer*[4] of a married woman, to confirm the words of Rabbi Jehoshua.' "

This passage is from the Mishnah, and therefore belongs to the older stratum of the Talmud. Ben Azai flourished at the end of the first and the beginning of the second century, and was a friend and contemporary of Rabbi Akiba who was a particularly zealous opponent of the Christians. When Ben Azai reported that he had found a book of pedigrees, in which it was stated that *so and so* (peloni) was of spurious birth, it is certainly probable that reference is to Jesus. Unless some well-known man were intended, there would no point in referring to him; and unless there had been some strong reason for avoiding his name, the name would have been given in order to strengthen the argument founded upon the case. For it is said that Ben Azai made his statement in order "to confirm the words of Rabbi Joshua." The matter in question concerned the definition of the notion of *mamzer*, a predicate which the Jews only too willingly ascribed to Jesus. That the passage refers to Jesus is admitted by the Jewish scholar J. Derenbourg (in *Revue des Etudes Juives*, III, 293, n. 3).

[2] It is interesting that the English translators of the Eighteen Treatises of the Mishna, rabbis De Sola and Raphall (London, 1845) have not translated this part of the fourth chapter. Why?

[3] The original reads *peloni*, and is one of the twenty-eight periphrastic titles of Jesus from Jewish writings, adduced by Eisenmenger in the second chapter of the first part of his *Entdecktes Judenthum*.

[4] I. e., a bastard, a predicate attributed by the Jews to Jesus.

UMI

THE ALLEGED CONFESSION BY THE MOTHER OF JESUS.

In the treatise Kallah 18^b we read the folowing: "A shameless person is according to Rabbi Eliezer a bastard, according to Rabbi Joshua a son of woman in her separation, according to Rabbi Akiba a bastard *and* son of a woman in her separation. Once there sat elders at the gate when two boys passed by; one had his head covered, the other bare. Of him who had his head uncovered Rabbi Eliezer said, 'a bastard!' Rabbi Joshua said, 'A son of a woman in her separation!' Rabbi Akiba said, 'A bastard *and* son of a woman in her separation!' They said to Rabbi Akiba, 'How has thine heart impelled thee to the audacity of contradicting the words of thy colleagues?' He said to them, 'I am about to prove it.' Thereupon he went to the boy's mother, and found her sitting in the market and selling pulse. He said to her, 'My daughter, if thou tellest me the thing which I ask thee, I will bring thee to eternal life.' She said to him, 'Swear it to me!' Thereupon Rabbi Akiba took the oath with his lips, while he canceled it in his heart. Then said he to her, 'Of what sort is this thy son?' She said to him, 'When I betook myself to the bridal chamber, I was in my separation, and my husband stayed away from me. But my paranymph [i. e., the bridegroom's best man] came to me, and by him I have this son.' So the boy was discovered to be both a bastard and the son of a woman in her separation. Thereupon said they, 'Great is Rabbi Akiba, in that he has put to shame his teachers.' In the same hour they said, 'Blessed be the Lord God of Israel, who has revealed this secret to Rabbi Akiba ben Joseph.'"

This famous discussion on bastardy, even when taken by itself is remarkable from the ethical point of view. Considering the strange ascription of an act of heartless perjury to Akiba as the means whereby he extorted the con-

PUBLIC LIBRARY.
No. 105.M7.....
DETROIT, MICH.

fession from the boy's mother, and the far more curious addition at the end of the passage which blesses the God of Israel for revealing "this secret" after the use of such questionable means, we ask: Can modern Jews still uphold the reputation of the Talmud or of its great authority, Akiba?

Considering the passage by itself, we see that neither the name of the son nor that of the mother is here mentioned. But the fact that use has been made of the story in the *Toldoth Jeshu* (ed. Hulreich, p. 22; ed. Wagenseil, p. 12; MS. Strassburg, ed. Krauss,[5] p. 39; MS. Vindobona, *ibid.*, p. 69) shows that it was regarded as having reference to Jesus. That Akiba lived about a century after Jesus, is of no account, since the Talmud abounds in anachronisms.

JESUS AND HIS TEACHER.

Whereas the New Testament knows nothing of Jesus having enjoyed the tuition of a rabbi, the Talmud *Sanhedrin*, 107*b* (and almost exactly in the same words *Sota*, 47*a*) narrates the following: "Our Rabbis teach, Ever let the left hand repel and the right hand invite, not like Elisha who repulsed Gehazi with both hands, and not like Rabbi Joshua ben Perachjah who repulsed Jeshu (the Nazarene) with both hands. What of Rabbi Joshua ben Perachjah? When Jannai the king killed our Rabbis, Joshua ben Perachjah [and Jeshu] fled to Alexandria in Egypt. When there was peace, Simon ben Shetach wrote to him: 'From me [Jerusalem] the city of holiness, to thee Alexandria of Egypt [my sister]. My husband stays in thy midst and I sit forsaken.' He came and found himself at a certain inn; they showed him great honor. He said, 'How beautiful is this Ascania!'[6] [Jesus] said to him, 'Rabbi,

[5] *Das Leben Jesu nach jüdischen Quellen*, Berlin, 1902.

[6] The word means both inn and innkeeper. Joshua uses it in the first sense, the answering remark implies the second meaning "hostess."

UMI

she has narrow eyes.' He said, 'Wretch, dost thou employ thyself thus?' He sent out 400 trumpets and excommunicated him. He [Jesus] came before him many times and said to him, 'Receive me.' But he would not notice him. One day he [i. e., Joshua] was reciting the Shema (i. e., the words: 'Hear, O Israel,' Deut. vi. 4 et seq.), he [i. e., Jesus] came before him. He was minded to receive him, and made a sign to him. He [i. e., Jesus] thought that he repelled him. He went and hung up a tile and worshiped it. Joshua said to him, 'Return.' He replied, 'Thus I have received from thee, that every one who sins and causes the multitude to sin, they give him not the chance to repent.' And the teacher [i. e., he who has handed down this tradition] has said, 'Jesus the Nazarene practiced magic and led astray and deceived Israel.'"[7]

In the Jerusalem Talmud *Hagigah, 77d,* the same story is related only that in place of Joshua ben Perachjah his contemporary Judah ben Tabbai is placed and that the name of Jesus is not mentioned, for which we read "one of his (Tabbai's) disciples." But whether the reading is Joshua ben Perachjah or Judah ben Tabbai we have here again one of those striking anachronisms for which the Talmud is famous. The event under King Jannai (i. e., Alexander Jannaeus) is historical. After the capture of the stronghold Bethome King Jannai (104-78 B. C.) had 800 Pharisees crucified. This crucifixion was the occasion of the flight into Syria and Egypt on the part of the Pharisees generally in the country, and among them Joshua ben Perachjah and Judah ben Tabbai. The question may be asked, how did the name of Jesus[8] come to be introduced into a story referring to a time so long before his own?

[7] This formal charge is also found in *Sanhedrin 43a,* where the words of "the teacher" are found.

[8] In the edition of Basel, 1578-81, and in all later ones, the censor of the press has expunged the name of Jesus, which is found in all the older editions of the Talmud.

Bearing in mind that the rabbis had extremely vague ideas of the chronology of past times, we may perhaps find the origin of the story in its Babylonian form in a desire to explain the connection of Jesus with Egypt. The connecting link may, perhaps, be found in the fact of a flight into Egypt to escape the anger of a king. This was known in regard to Joshua ben Perachjah, and the Gospel (Matt. ii. 13 et seq.) records a similar event in regard to Jesus. There may be some other details in the life of Jesus which the rabbis had in view when they remodeled the story to suit their purpose. Hence, in rejecting the date, it is not absolutely necessary to reject the whole of the Babylonian version as entirely devoid of every element of genuineness. Again, as to the lateness of the Babylonian version, it is to be observed that the Gemara quotes from an earlier source or tradition of the story, as can be seen from the closing words of the Talmud passage (*Sanhedrin* 107*b*) given above.

JESUS A MAGICIAN.

Whereas the Toldoth Jeshu attributes the miraculous power of Jesus to the Shem (i. e., the Tetragrammaton or Ineffable Name) which he stole from the Temple at Jerusalem by a strange device, the Talmud knows nothing of this robbing of the Shem from the Temple, but records that Jesus brought magic out of Egypt.[9] The passages referring to it are:

1. Tosephta[10] Shabbath XII: "He who upon the Sabbath cuts letters upon his body is, according to the view of Rabbi Eliezer guilty, according to the view of the sages he is not guilty. Rabbi Eliezer said to the sages: 'Ben Stada surely learned sorcery by such writing.' They re-

[9] Egypt was regarded as the special home of magic, an opinion expressed in the Talmud Kiddushin 49*b*: "Ten measures of sorcery descended into the world, Egypt received nine, the rest of the world one."

[10] That is "supplement to the Mishna"; best edition by Zuckermandel, (Pasewalk, 1880), p. 126.

plied to him: 'Should we in any wise on account of a fool destroy all reasonable men?' "

2. Jerusalem Shabbath 13*d*: "He who scratches on the skin in the fashion of writing is guilty, but he who makes marks on the skin in the fashion of writing is exempt from punishment. Rabbi Eliezer said to them: 'But has not Ben Stada brought magic spells out of Egypt in this way?' They answered him: 'On account of one fool we do not ruin a multitude of reasonable men.' "

3. Shabbath 104*b*: "It is a tradition that Rabbi Eliezer said to the sages: 'Did not Ben Stada bring spells from Egypt in a cut which was upon his flesh?' They replied, 'He was a fool, and they do not bring a proof from a fool.' "

It has already been shown above that Ben Stada denotes Jesus. In the passages before us he is charged with bringing magical charms from Egypt concealed in an incision in his flesh. The charge that he was a magician is no doubt based on the belief that he did many miracles, a belief which found ample support in the Gospel records. To say that Jesus learned magic in Egypt, the special home of magic according to the Talmud Kiddushin 49*b* already referred to, is to say that he was a great magician, more powerful than others. That he had something to do with Egypt we have also seen above in the passage which makes him a disciple of Joshua ben Perachjah. As to the manner in which he is alleged to have brought with him Egyptian magic, a curious explanation is given by Rashi, the Talmud commentator on Shabbath 104*b* to the effect that "the Egyptian magicians searched every one who quitted the land of Egypt, whether he was taking any books of magic with him, in order that the magical art (namely, the Egyptian) might not come into other countries." Now since Jesus could not take them away in writing, he concealed them in the manner described, or perhaps tatooed magical signs on his flesh.

JESUS AN IDOLATER.

Jesus is not only a fool but also an idolater. In the Talmud Sanhedrin 103*a* we read on the passage Ps. xci. 10, "There shall no evil befall thee," that it means that evil dreams and bad fantasies shall not vex thee; on "Neither shall any plague come nigh thy tent," that it means that thou shalt not have a son or disciple who burns his food publicly like Jeshu (i. e., Jesus) the Nazarene.[11]

The last clause is also found in Talmud Berachoth 17*b*. The authority who quotes this passage is Rabbi Hisda, a Babylonian who lived A. D. 217-309. He quotes it in the name of Rabbi Jeremiah bar Abba, who was his contemporary, and apparently of about the same age. As to the term "to burn his food publicly," lexicographers are of different opinions. Dalman says that this means "to renounce publicly what one has learned." Laible thinks that the term is "a contemptuous expression for the public offering of sacrifice to idols. That the Christians in their assemblies offered sacrifice to idols was as firmly the opinion of the Jews of olden time as it is that of many of the present day. Naturally, therefore, it was concluded that Jesus must have commenced it." May be it refers to the fact that Jesus went and taught the people publicly—the poor, the outcast, the oppressed, the sinners, the publicans, in a word the unpurified people, with whom a disciple of a rabbi ought not to associate. But whatever the meaning, certain it is that in the eyes of the rabbis Jesus was a heterodox, who according to Talmud Sanhedrin 43*a* and 107*b* "corrupted and seduced Israel."

THE CLAIM OF JESUS DENIED.

In the Jerusalem Talmud Taanith 65*b* we read with reference to Num. xxiii. 19: "Rabbi Abahu said, If a

[11] So in all the older editions and the manuscripts.

man shall say to thee, 'I am God,' he lies; if he says, 'I am the son of man,' he shall rue it; 'I will go up to heaven' (to this applies Num. xxiii. 19) he saith, but shall not perform it."

That the passage refers to Jesus there can be no possibility of doubt. This Rabbi Abahu, who lived in Cæsarea at the end of the third and the beginning of the fourth century, seems to have largely engaged in controversy with Jewish Christians. According to Abahu any one who says that he is God and at the same time designates himself as Son of Man—and this no man save Jesus has ever done —is a liar.

The import of the testimony of Jesus to Himself is mentioned also in the Midrash Pesikta Rabbathi (ed. Friedmann, 1880), fol. 100*b*: "Rabbi Hia bar Abba [about 216 A. D.] said: 'If the son of the harlot shall say to thee, There be two Gods, answer him, I am He of the sea, I am He of Sinai.' Rabbi Hia bar Abba said, 'If the son of the harlot shall say to thee, There be two Gods, answer him, It is here (Deut. v. 4) written not *Gods* but *the Lord* hath spoken with you face to face.'"

That God has a son, and that for this reason there are two Gods, passes here for the teaching of the harlot's son, wherein the reference is clear, namely to Jesus.

An amplification of Abahu's work, given above, is found in the Midrash Jalqut Shimoni (also Midrash Jelammedenu) on Num. xxiii. 7, where we read that Rabbi Eleazar ha-Qappar said that "God gave strength to his (Balaam's) voice, so that it went from one end of the world to the other, because he looked forth and beheld the peoples that bow down to the sun and moon and stars, and to wood and stone, and he looked forth and beheld that there was a man, son of a woman, who should rise up and seek to make himself God, and to cause the whole world to go astray. Therefore God gave power to his

voice that all the peoples of the world might hear, and thus he spake,'Give heed that ye go not astray after that man, for it is written (Num. xxiii. 19), 'God is not a man that he should lie,' and if he says that he is God he is a liar, and he will deceive and say that he departeth and cometh again in the end, he saith and he shall not perform. See what is written (Num. xxiv. 23): 'And he took up his parable and said, Alas, who shall live when he makes (himself) God!' Balaam intended to say, 'Alas, who shall live of that nation which heareth that man who hath made himself God."—

Now Eleazar ha-Qappar, who is reported to have said all this, was earlier than Abahu, for he died about 260 A. D. Whether all is to be taken as Qappar's work, we know not. At all events we have here a naive prophecy after the event, which makes Balaam quote his own words (Num. xxiii. 19) as scripture. One thing however is certain: Jesus is here referred to more fully than in the shorter saying of Abahu.

BALAAM-JESUS.

In Mishna Sanhedrin x, 2, we read: "Three kings and four private men have no part in the world to come. The three kings are Jeroboam, Ahab and Manasseh....the four private men are Balaam, Doeg, Ahitophel and Gehazi."

This passage belongs to the famous chapter of the Mishna, entitled *Chelek,* because it commences by saying that "all Israel have part (*chelek*) in the world to come," and then enumerates the exceptions. The three kings, Jeroboam, Ahab and Manasseh are all mentioned in the Old Testament as having introduced idolatry, perverted the true religion. The immediate connection of the four private persons arouses the conjecture that they were condemned for the same offense. This conclusion is strength-

ened by the fact that the preceding paragraph of the Mishna (x, 1)in this chapter excepts from the privilege of the world to come according to Rabbi Akiba also such a person "who reads in external books and who whispers over a wound, and says, None of the diseases which I sent in Egypt will I lay upon thee, I the Lord am thy healer." Now the external books, according to the Gemara upon this passage (fol. 100*b*) are the *Siphre Minim,* i. e., the books of the Jewish Christians or Christians generally, which books by way of caricature Rabbi Meïr (130-160 A. D.) calls *awen gillayon* (literally, margin of evil) and Rabbi Jochanan (Meïr's contemporary) calls *awon gillayon* (i. e., blank paper of sin)—thus in Talmud Shabbath 116*a* (MS. Munich).

The words "who whispers over a wound," refer to the miraculous cures of the Christians.

The combination of Balaam with Doeg, Ahitophel and Gehazi is certainly extraordinary. Balaam was not an Israelite, and therefore could not logically be included in a list of exceptions to a rule which only affected Israelites. It is evident that Balaam here does not mean the ancient prophet of Num. xxii et seq., but some one else for whom that ancient prophet could serve as a type. From the Jewish point of view there was considerable likeness between Balaam and Jesus. Both had led the people astray; and if the former had tempted them to gross immorality, the latter, according to the Rabbis, had tempted them to gross apostasy. This was the great charge against Jesus, that "he practised magic and deceived and led astray Israel." If it be true that Balaam stands for Jesus, then it is reasonable to suppose that Doeg, Ahitophel and Gehazi stand for the names of some other persons who had fallen under severe Rabbinical displeasure. Who they were precisely we have now no means of discovering, and the supposition that they refer to Peter, James and John, or Peter,

Judas Iscariot[12] and Paul may be possible. However this may be, the rabbis were convinced that the disciples of Balaam *en bloc* would inherit Gehenna, as we read in Aboth v, 19: "The disciples of our father Abraham enjoy this world and inherit the world to come, as it is written (Prov. viii. 21) 'That I may cause those that love me to inherit substance, and that I may fill their treasuries.' The disciples of Balaam the impious inherit Gehenna and go down into the pit of destruction, as it is written (Ps. lv. 24): 'But thou, O God, shalt bring them down into the pit of destruction; bloodthirsty and deceitful men shall not live out half their days:' "

And if there should by any chance be still the slightest hesitation in the mind of the reader that Balaam in these passages is identical with Jesus, the following passage should forever set his mind at rest.

THE AGE OF BALAAM (JESUS).

In the Babylonian Talmud Sanhedrin 106*b* we read thus: "A certain heretic (*min*) said to Rabbi Hanina, 'Have you ever heard how old Balaam was?' He replied, 'There is nothing written about it. But since it is said, Bloodthirsty and deceitful men shall not live out half their days (Ps. lv. 23), he was either thirty-three or thirty-four years old.' He (the heretic) said, 'Thou hast spoken well. I have seen the chronicle of Balaam in which it is said, Balaam, the lame, was thirty-three years old when the robber Phinchas killed him.' "

Rabbi Hanina lived at Sepphoris and died 232 A. D. There seems to be no apparent reason why a Christian (a *min*) should have asked him as to the age of the ancient Balaam. He might well have inquired about the age of Jesus. It would seem, however, that the Christian was

[12] Judas Iscariot would answer to Doeg the Edomite, who betrayed David (1 Sam.xxii. 9).

not asking for information, but had a desire to find out whether the rabbi knew anything about Jesus. For he confirmed the rabbi's answer by facts known to himself. The "Chronicle of Balaam" probably denotes a Gospel, though none of the known Gospels states in so many words that Jesus was as much as thirty-three years old. If, however, it was believed that his ministry lasted three years, and that he was "about thirty years old" when he began to preach, the statement of the Christian is sufficiently borne out, though not verbally correct. Rabbi Hanina must have had fairly good grounds for his opinion as to the age of Jesus, or he would not have quoted a text which would only apply to the case of a man about thirty-three or thirty-four years old.

As to Phinchas the robber or "Pinchas Listāāh" who is said to have killed Balaam, it is difficult to understand why this worthy, who is mentioned in Num. xxv. 23 et seq. as having led an army against the Midianites and slain their kings together with Balaam with the sword, and this at Moses's command, should be called "the robber." Some Jewish writers see in Pinchas Listāāh a corruption of Pontius Pilate. The corruption, we admit, is a somewhat violent one; but that a Jew should call Phinchas a robber, being, as he was, a highly honored hero of tradition, must certainly be surprising. There is no doubt that under this mention of Pinchas Listāāh there lies concealed a reference to Pontius Pilate.

THE TRIAL OF JESUS.

1. In the Mishna Sanhedrin X, 11, we read: "In regard to all who are worthy of death according to the Torah, they do not use concealment against them, except in the case of the deceiver. How do they deal with him? They put two disciples of the wise in the inner chamber, and he sits in the outer chamber, and they light the lamp so that

they shall see him and hear his voice. And thus they did to Ben Stada in Lud; two disciples of the wise were chosen for him, and they (brought him to the court of justice) and stoned him."

2. In the Jerusalem Recension VII, 16 (fol. 25*c*, *d*) we read: "The deceiver; this denotes a private man. Not a sage? No. From the time he deceives he is no longer a sage. And from the time he is deceived he is no longer a sage. How do they deal with him to work craftily against him? They conceal (in his case) two witnesses in the inner chamber and make him sit in the outer chamber, and they light a lamp over him that they may see him and may hear his voice. Thus did they to Ben Stada in Lud, and they concealed in his case two disciples of the wise, and brought him to the court of justice and stoned him.

3. The Babylonian Gemara Sanhedrin 67*a* has the following version of this incident: "For it is a tradition that in regard to the rest of all who are worthy of death according to the law, they do not use concealment except in this case (i. e., of the deceiver). How do they deal with him? They light a lamp for him in the inner chamber and set witnesses in the outer chamber, so that they may see him and hear his voice, but he does not see them. And one says to him, 'Say to me what thou saidst to me in private.' and he says it to him. And another says to him, 'How shall we forsake our God who is in heaven, and practice false worship?' If he repents, it is well. If he says, 'Such is our duty and thus it becomes us to do,' the witnesses who hear him from outside, bring him to the court of justice and stone him. And thus they did to Ben Stada in Lud, and they hung him on the eve of the Passover."

That the case described in these passages refers to Jesus (called Ben Stada), who was also charged with deceiving

the people, is clear. It is also clear that at an early period there was a tradition that the condemnation of Jesus had been obtained by the fraudulent means described above. There can be no doubt that in these passages we have here only scanty remnants of a tradition about that trial, combined perhaps with hearsay information derived from Christians. Renan in his *Life of Jesus* (chap. 24) believes that the New Testament account of the Trial of Jesus must be supplemented by these Talmudic notices. But an equally good, if not better authority, Keim (*Jesus of Nazara* VI, 47n.) says that there is no ground for correcting the Gospel account by the help of the Talmud. Rather it is the Gospel account which throws light upon the Talmudic tradition. From the Gospel story are derived the two witnesses (Matt. xxvi. 60. In Mark xiv. 56, 57, several witnessed are mentioned). The Gospel speaks of "false" witnesses, and this is perhaps the origin of the Talmudic assertion that the witnesses were concealed in order to entrap the accused. The mention of the outer and inner chamber recalls Matt. xxvi. 69, where it is said that "Peter was sitting without in the court" while the trial was going on within the house of the High Priest. The lighted lamp may have been suggested by the mention of the fire kindled in the outer court (Luke xxii. 55). And finally the statement that the accused was carried to the court of justice, may have its origin in the fact that there was, according to the Gospels, a second sitting of the council after the one at which the witnesses had been present (Mark xv. 1). The Talmudic tradition differs from the Gospel in saying that the trial took place at Lud (Lydda), and that Jesus was hung on the eve of the Passover. Of this we shall speak further on. But all tends to show that the Talmud has preserved only a very vague and confused recollection of Jesus, whose name was doubtless held in abhorrence as that of a dangerous heretic and deceiver.

UMI

THE EXECUTION OF JESUS.

We read Sanhedrin 43*a*: "And it is tradition: On the eve of the Passover they hung Jeshu [the Nazarene]. And the crier went forth before him forty days (saying), '[Jeshu the Nazarene] goeth forth to be stoned, because he hath practiced magic[13] and deceived and led astray Israel. Any one who knoweth aught in his favor, let him come and declare concerning him. And they found naught in his favor. And they hung him on the eve of the Passover. Ulla said, 'Would it be supposed that [Jeshu the Nazarene] a revolutionary, had aught in his favor?' He was a deceiver, and the Merciful (i. e., God) hath said (Deut. xiii. 8), 'Thou shalt not spare, neither shalt thou conceal him.' But it was different with [Jeshu the Nazarene], for he was near to the kingdom."[14]

In this passage we are told that Jesus was hung. With this must be combined the evidence of the passages given in the former section that he was stoned. The connection between the two statements is that Jesus was stoned, and his dead body then hung upon a cross. This is clear from the Mishna Sanhedrin vi, 4: "All who are stoned are hung,

[13] It is certainly strange that Jesus was charged with having practiced magic, whereas magical skill was one of the qualifications necessary for a member of the Sanhedrin. Thus we read in treatise Sanhedrin 17*a*: Rabbi Jochanan says, none were allowed to sit in the Sanhedrin, who were not men of stature, men of wisdom, men of good appearance, aged, skilled in magic, and acquainted with seventy languages, so that the Sanhedrin might not be obliged to hear through an interpreter.—That this statement is in opposition to Deut. xviii. 10-12 makes no difference with the rabbis. The commentary indeed tells us, that this magical skill was acquired in self-defence "in order to kill the magicians who trusted in their magical arts to deliver them out of the hands of the tribunal." But this explanation does not mend the matter. Magic is a thing absolutely unlawful and expressly forbidden by God. But it may well be doubted whether the members of this great council confined their magical exercitations to the killing of magicians. We find elsewhere, that the rabbis at least made other magical experiments, and have even recorded the means which they employed, for the benefit of posterity. We refer to Talmud Berachoth 6*a*, where the people and all Israel are instructed in the means to see demons. The passage being too silly, we refrain from giving it.

[14] The whole of this passage is expunged from the later editions. It is given here on the authority of the MSS. and early editions set forth by Rabbinowicz. The words in [ ] are from MSS.

according to Rabbi Eliezer. The sages say, None is hung except the blasphemer and he who practices a false worship." The corpse was hung to a cross or else to a single beam, of which one end rested on the ground, the other against a wall (same Mishnah). The Gospels, of course, say nothing about a stoning of Jesus, and the Talmudic tradition is probably an inference from the fact that he was known to have been hung. The inference would be further strengthened by the application of the text, Deut. xxi. 23, "He that is hanged is accursed of God," a text which Paul had to disarm in reference to Jesus (Gal. iii. 13). The Talmud knows nothing of an execution of Jesus by the Romans, as modern Jews claim, but makes it wholly the act of the Jews.

What is meant by the herald going forth during forty days before the death of Jesus, is hard to tell. The herald is, of course, fictitious; the number forty may have its origin in the Gospel. The phrase that Jesus was "near to the kingdom," Laible interprets as referring to the "Roman authorities," which would explain the hesitation of Pontius Pilate to put Jesus to death. We rather prefer the suggestion that the reference is to the supposed Davidic descent of Jesus, a suggestion made by the late Professor Delitzsch in his *Jesus and Hillel* (3d ed., 1879) where he says on page 12, note: "Mary is also called in the Talmuds a daughter of Eli, and Jesus is called (Sanhedrin 43*a*) 'related to the royal house (of David).' "

JESUS IN HELL.

In the Talmud Gittin 56*b*, 57*a*, we read: "Onkelos bar Kalonikos, nephew of Titus, desired to become a Jew. He called up Titus by necromancy. He said to him, 'Who is honored in this world?' He replied, 'Israel.' 'What about joining them?' He replied, 'Their words are many and

thou canst not fulfil them. Go, join thyself to them in this world and thou shalt become a leader, for it is written (Lam. i. 5), "Her adversaries have become the head." Every oppressor of Israel is made a head.' He said to him, 'What is thy punishment?' He replied, 'That which I have determined for myself. Every day my ashes are collected and I am judged; then I am burnt and the ashes scattered over seven seas.'

"He called up Balaam by necromancy. He said to him, 'Who is honored in this world?' He replied, 'Israel.' 'What about joining them?' He replied (Deut. xxiii. 6), 'Thou shalt not seek their peace or their prosperity all thy days.' He said to him, 'What is the punishment of this man?' He replied, 'By boiling pollution.'

"He called up Jesus by necromancy. He said to him, 'Who is honored in this world?' He replied, 'Israel.' 'What about joining them?' He replied, 'Seek their good, seek not their harm. Every one who injures them, (it is) as if he injured the apple of his eye.' He said, 'What is the punishment of this man?' He replied, 'By boiling filth.' For a teacher has said, 'Every one who mocks at the words of the wise is punished by boiling filth.' Come and see the difference between the sinners of Israel and the heathen prophets!' "

The object of the gruesome story contained in this passage is to show the fate of the three chief enemies of Israel, i. e., Titus, Balaam and Jesus. And although Jesus is made to regard the Jews as the chosen race, the specially beloved, the apple of Yahveh's eye, yet his punishment seems to be the severest. Whatever that punishment was we know not. At any rate it expresses a hatred towards the most hated of all hated men.

The information which we derive from the Talmudic notices of Jesus is very little if any at all. They add nothing new to the authentic history of Jesus, as contained in

the Gospels. In general, though not in detail, they serve to confirm the Christian tradition, by giving independent, and indeed hostile evidence that Jesus of Nazareth really existed, a fact which has by some been called in question. But if, beyond this, the Talmudic Jesus-tradition has no value for the history of Christianity, it shows the attitude of Judaism as represented by its leaders, towards Jesus. He is the deceiver, the sorcerer, the apostate, the "Sinner of Israel"; his birth Jewish contempt blackened into a disgrace, and his death is dismissed as the mere execution of a pernicious criminal. And thus it is to this day. To understand Jesus and his religion Judaism must divest itself of Rabbinism. Not the Talmud, but the New Testament is the real source for the history of Jesus.

BERNHARD PICK.

NEWARK, N. J.

THE NATURE OF LOGICAL AND MATHEMATICAL THOUGHT.

INTRODUCTORY.

HUMAN thought is dominated by methods based upon a principle which in its various applications is called reason, and the total set of the rules of reason is called logic. In the same way all computations, and further all space-conceptions, measurements of distance and of direction, depend upon a science which has much in common with logic and has received the name "knowledge-lore" or "mathematics."

Experience has shown that the accomplishments of both logic and mathematics are most marvelous. They are the woof of the web in the fabric of all the sciences, and human civilization is their most palpable product. In fact we may say that man himself, especially the scientific thinker, is nothing but reason (viz., logic and mathematics) incarnate. All that distinguishes man from brute creation consists in his ability to think with definite methods, to be logical and exact in measuring and counting.

Now it is strange that the nature of man's rationality is by no means universally recognized. Opinions vary greatly concerning its foundation and its origin, and this divergence has come out most plainly in a new development of mathematical thought which has produced peculiar systems of mathematics differing from the traditional Euclidean system. There is still missing, however, a new

UMI

system of logic which would be contradictory to the logic of Aristotle.

The revolution against the old views began with an attack on the axiom of parallel lines; and the idea that through any point C, there ought to be one and only one straight line parallel to a given straight line AB, has been set aside to make room for a higher and more general mathematics, a pangeometry, where the Euclidean assumption would be only one special case among other possibilities. Other new systems have indeed been developed in which Euclid's parallel postulate is set aside as unproven, and as a result the view has been commonly accepted that other non-Euclidean geometries are possible.

Mathematics is at present dominated by a tendency which may be called experimentalism. The mathematician hungers for facts, for a basis in the realm of concrete sense-experience. He envies his brethren the naturalists, whose methods since the days of Darwin have enjoyed an enormous boom. He has been living for centuries in a domain of pure thought, and he wants now to stand on the ground of actuality. The natural sciences have attained wonderful results in the shape of inventions and discoveries, and some mathematicians feel that they are left behind in the race and so they are making vigorous efforts to emulate the naturalist method of investigation.

This hankering for facts in the domain of mathematics is in our opinion an aberration. Mathematics is a creation of pure thought. It is built up in the domain of anyness—a product of abstraction. Questions as to the nature of actual (i. e., objective) space, whether it be Euclidean or non-Euclidean, homaloidal or curved, homogeneous or heterogeneous, three-, four-, or *n*-dimensional, are all beside the mark. Mathematicians of great repute, who at the same time are masters of all the details of their science, have raised questions such as these: "Will not a straight

line finally, after billions of miles (or perhaps at a distance of billions of light years) return into itself?" or "May not the sum of the angles of a plane triangle, if only measured in cosmic proportions sweeping through the stellar heavens, prove to be a little more or a little less than 180 degrees?" or, "Are the opposite angles in a parallelogram really equal?" or "Is objective space such as we think it or is it different? Is space Euclidean or non-Euclidean? Is it tri- or four- or many-dimensional?" All these and kindred problems prove that those who propose them,—I say so with all deference to their learnedness and yet with perfect assurance—do not understand anything of the foundations of mathematics.

According to my conception of mathematics, we have created the plane, and in plane geometry the straight line is straight and remains straight into infinity, the right angle is a right angle wherever it may be constructed in a plane, and the angles of a plane triangle measure exactly 180 degrees, nothing more, nothing less. There is no approximation; everything is exactly so. Such is the nature of mathematical thought which, in this respect, is different from the facts of the natural sciences. In the natural sciences our observations and measurements are never perfectly exact; they are always approximations.

The natural sciences deal with particulars, and generalized statements have been gained by induction from an observation of several or many particular experiences. But logic and mathematics are sciences of pure form and their productions are mental constructions which are rigidly and unequivocally determined, and there is no approximation about their truth. I may add here that as there are no mathematical planes and lines so there are no syllogisms in the objective world of fact, but there are uniformities for the tracing of which logical rules are serviceable, and in the domain of logic the syllogisms are

UMI

as rigid as are the propositions of Euclid in plane geometry.

To the new-fangled non-Euclideans and to adherents of the New-Science conception this statement may appear antiquated and old-fashioned, but a close inspection will prove that science still stands on the old foundations, and though in the course of modern development new and broader viewpoints have been gained, science will after all be found to remain on the Rock of Ages, on that irrefragable consistency of natural events which can be formulated in the so-called natural laws and finds its noblest development in the rationality of the human mind, viz., in those eternalities which are ultimately nothing but the consistency of thinking, the consistency of doing, the consistency of being. The author has published two books on this most important problem, *Kant's Prolegomena* (being a critical discussion of the Kantian solution) and *The Foundations of Mathematics*; and he wishes here to present a brief recapitulation of his views and add some comments on conceptions which differ from his own.

THE AUTHOR'S POSITION.

The belief in a consistency of existence is first a mere faith, based upon an instinctive apprehension of law underlying all regularities; but this faith proves the more reliable the deeper we penetrate into the nature of being.

A condition of uniformities which admits a possibility of formulating them in natural laws is called in German, *Gesetzmässigkeit,* and this has as yet no equivalent in English. We propose to call it "lawdeterminedness" or simply "lawdom,"* and would define lawdom as a state of things in which all events take place according to general rules,

* The word "lawdom" is formed in analogy to kingdom, freedom, wisdom, Christendom. *Dom* is derived from the same root as *doom,* "judgment," and means in kingdom the dominion of a king; in wisdom, the prevalence of the wise; in freedom, the sway of the free; and in lawdom, it means a condition determined by law.

viz., the laws of nature. In this sense we say that the consistency of natural phenomena manifests itself as lawdom.

Every contradiction is a problem and every solution of a problem becomes a renewed justification of our belief in the consistency of existence. This belief appears for a time as a divine revelation and finally becomes the assured result of science. If there were no consistency there would be no science, reason would be a mere coincidence of haphazard regularities, and a trust in the efficiency of reason should be branded as a vagary of deluded dreamers.

The very existence of reason is an evidence that the universe is consistent throughout, and human reason is an instinctive comprehension of this most remarkable feature of existence, while science is simply the methodical application of reason.

This resumé sounds very simple, yet sometimes it is difficult to state and comprehend simple truths. We shall have to grant that simple truths stand in need of elucidation, for in the infinite manifoldness of actual existence they are rendered quite complex, and thus it happens that great thinkers encounter many difficulties which can be surmounted only by a most scrupulous exactness.

While experience and experiment can not settle the problems as to the nature of mathematical space, we must grant that there is one great truth in the tendency of modern mathematics. It is this, that mathematics is not absolutely independent of experience. Though mathematics is a purely mental construction, the method of its construction is derived from experience. In other words, though mathematics is, in the terminology of Kant, *a priori*, our *modus operandi* is a function which we have procured by abstraction from our activity evinced in the domain of the *a posteriori*. We cancel in thought everything particular which comprises all things concrete, be they of matter or

energy, and retain only our mental faculty of doing something, including a field of action implied by the possibility of moving about. This field of action with its absolute absence of all particularity is characterized by generality. In other words, it is the domain of anyness.

Accordingly we do not start in mathematics with nothing, nor do we go about our business blindly. In arithmetic we operate by taking a step and repeating it again and again. Thus we posit a unit, then we proceed to posit another unit and another, and each unit is the same as all the rest. We count them and operate with their sums. Such is arithmetic or the science of numbers.

When we bear in mind that mathematics is a mental construction we will readily understand that sums in arithmetic are products of synthesis. Every number is the result of an addition, and addition is no mere analysis of the idea of number; it partakes of the synthetic character and becomes possible only through the procedure of positing new units and summing up the total result.

Kant was astonished to find that even the most simple arithmetical calculation (such as $8+5=13$) was not the result of a mere analysis of the numbers implied, but was of a synthetic nature. Analytical judgments do not teach us new truths; they only render the ideas we have clearer and more definite, while synthetic operations increase our stock of knowledge. This puzzled him, for according to his nomenclature all mathematical, arithmetical and logical propositions were *a priori*, and all *a priori* propositions were quite commonly (though erroneously) assumed to be purely analytic. So he came to the conclusion that man's faculty of making *a priori* constructions constituted in itself a source of positive knowledge,—of knowledge that could be increased and amplified without resorting to sense-experience.[1]

[1] I use the term sense-experience on purpose so as to limit the meaning

By *a priori* knowledge Kant understands all that knowledge which is presupposed in experience of any kind. When the chemist analyses some compound and finds in his retorts 87% of its mass, he concludes that he lost 13%; he does not assume that during the process 100 particles have shrunk into 87, or that 13% have vanished into nothing. They are lost to him but have not been annihilated. In the same way all mathematical and logical propositions are relied upon. They are trusted above all experience, and if we make an experiment the result of which contradicts them (or seems to contradict them), we doubt our observations and distrust our experiment. We seek the fault in our notion of the facts in question, not in the principles of reason. We may distrust our calculations and our arguments, but we never doubt the reliability of mathematics and logic. If an astronomer watches a comet, and determines three stations of its course by observation, he can map out a curve which is analogous to its path of motion, and in the same way all the formal sciences furnish us with a key that will unlock to us the mysteries of objective existence. This state of things, the agreement of our purely formal thought-construction with the laws of nature, is a most wonderful coincidence and it puzzled Kant to such an extent as to make of him an idealist, but the problem is solved if we bear in mind the "anyness" which characterizes our purely formal constructions. If consistency dominates both objective existence and our thought, both will be analogous.

How do we produce this anyness?

In geometry we begin with mapping out our field of operation. First we ignore everything actual or concrete; both matter and force are treated as if they were nonexistent and all that is left is motility. We can move in

of the term and to avoid the mistake resulting from the looseness of its use in Kantian nomenclature. Cf. the author's *Fundamental Problems,* pp. 26 ff., especially 30.

any direction and everywhere without end. Suppose we spread out in all directions at once by swelling up, or by spreading like light from a source of luminescence, we would cover the entire possibility of our scope of motion. In such a spread of motion we call a path of greatest intensity corresponding to a ray of light a straight line. Now we cut space in two and call the boundary between the two halves a surface. If the cut has been made evenly, which means by a ray and along a ray, i. e., by a straight line, which is a line that follows the path of greatest intensity,[2] we can flop the surface upon itself and we call it "a plane."

As a visible representation of the plane we use a sheet of paper which when folded upon itself produces the straight line. We use the folded sheet as a ruler and operate with it. We lay down units of length (feet and inches, or meters and centimeters) for the sake of measuring lines. Then we draw straight lines in different directions and make them intersect. Their products are angles. We make three lines intersect and call the figure thus created a triangle.

Further on we fold the creased paper upon itself and name the corners right angles. The plane, the straight line, the right angle are boundary conceptions which are useful because they are unique. There are innumerable curves, but only one straight line; there are innumerable obtuse and acute angles, but only one right angle; and thus these boundaries, these products of halving, will serve us as standards of reference.

Our next step is the creation of a curve that by its simplicity would possess the advantage of uniqueness. So, we draw a circle on our doubly folded sheet of paper from the point where the two creases meet. Following historical tradition which can be traced back to the sages of

[2] Compare *Foundations of Mathematics*, pp. 57-58.

ancient Babylon, we divide the whole circle into 360 degrees and we may remember here how their mathematical instinct was guided and influenced by some facts of observation. They rounded off the number of days from 365 to 360,[8] and divided the course of the sun on the ecliptic into twelve mansions of 30 degrees each, corresponding to 12 double hours per day.

The next step in geometrical constructions will be the transfer of angles and the drawing of two straight lines running in the same direction. We call them parallels. When two parallels are crossed by a third straight line, we investigate the nature of the eight angles thus produced.

We continue to operate by setting ourselves a series of tasks, and in doing so we can follow Euclid's propositions in their regular order by dealing with three intersecting lines and then with the circle and other figures. In this way we build up plane geometry without axioms or assumptions through our own operations, and we remain conscious of the method by which we came into possession of the straight line, the right angle, the parallel, etc.

There is nothing actual about our operations. All our achievements are purely mental; they lack concrete reality. There is no matter, no force in our constructions, and yet they are not nothing. The path of our motion is a line, and where two lines cross we have a point. A point is no concrete thing, yet it is not a nonentity; it is a locus in the field of our motion, a spot the position of which is definitely determined on either of the crossed lines. We do not find a plane anywhere in actual life, we construct it; and in the same sense straight lines and right angles are the products of our construction.

[8] The difference was made up every sixth year by the introduction of an intercalary month—the month of the raven, and it is noteworthy that the number 13 as well as the symbol of the thirteenth monh, the raven, have remained omens of ill luck to this day.

Nothing proves so well that our space-conception is in Kant's sense *a priori* as the possibility of non-Euclidean geometries. There is only one rule to guide our operations, consistency, and since particularity of any kind has been banished, the same operation will always and everywhere produce the same result. We operate in an absolutely empty field and our constructions are solely determined by the nature of our operations. All we have to do is to note the consequences of our transactions.

We might have constructed another field for our operations, for instance, the surface of a globe; and if we had done so from the start, our products would have been different. Straight lines would have become impossible and lines analogous to Euclidean straight lines would be largest circles. They are "shortest" or "straightest" lines; not truly straight in the Euclidean sense, they are the straightest lines possible. While two straight lines in the plane never enclose a space, two straightest lines on the sphere always enclose a space; and while the former intersect in one point (or if they are parallel not at all), the latter always intersect at two points and these two points are antipodal.

The construction of spherical geometry is quite simple and it is as easily pictured in visible figures as plane geometry; but there are other geometries possible, less simple and more difficult to describe or to render representable. Each one of them possesses its own characteristics and theoretically considered all of them are equally legitimate. They are all mental constructions. They are all based upon the principle of consistency and obey the general laws of logic, for if they did not recognize consistency, our operations would end in chaos.

The systems of Euclid, of Bolyai, of Lobatchevsky and others, including 4- or n-dimensional manifoldnesses, are *a priori* on the same footing. The difference comes in

when they are applied to practical purposes, and here the Euclideans have after all the advantage. The non-Euclideans make up for it by an enthusiasm as strong as the zeal of religious devotees which on the one hand deserves our admiration while on the other it has a humorous aspect.

That logic and mathematics come from the same root must have been felt by Euclid and his school, for what they call "common notions" are formulations of logical principles, while the description of the characteristics of space are laid down in the definitions and postulates. But the significance of the kinship between these two sciences, it appears, was first felt by Kant who may have been guided in this by his great contemporary Lambert.

Since the time of Kant, both logicians and mathematicians have felt the need of investigating the nature of thought-operations and of broadening the concepts of logic in a similar way as the metageometricians endeavor to construct a pangeometry which would be independent of our conception of Euclidean space. The first classical work which broadened the traditional logic was written by George Boole under the title *The Laws of Thought*, and since then logicians have felt the insufficiency of Aristotle's logic and the need of deciphering the nature of thought in its operations. They attempted to transfer the accomplishments of mathematics upon logic, and to exhibit the function of reason in formulas, or in graphic presentations, or in algebraic notations. Workers in this line are Ernst Schroeder, Charles S. S. Peirce, Giuseppe Peano, Bertrand Russell, and Louis Couturat. It is a new branch of scientific endeavor and we may expect results of great interest, yea even of far-reaching importance.

The writer's opinion is that labors of this kind constituting the new mathematics and the new logic are quite legitimate. They will widen our horizon but they do not (and never will) reverse, antiquate, or abolish the assured

accomplishments of the past. Neither Bolyai nor Lobatchevsky upsets Euclid and none of the modern logicians will ever set aside Aristotle.

NON-ARISTOTELIAN LOGIC.

Now, it is possible to imagine a fairy-tale world where our scientific conception of cause and effect could be crossed by a causation of miracle. In such a world the magician's word would be endowed with an energy unknown in physics, but it would remain a world governed by law, and the rule of consistency would not be upset. Every effort would presuppose a cause and causation would still be dominated by law. The purely formal rules of Aristotelian logic would not be upset thereby. The mill remains the same even if the grist is changed. We would have law-determinedness or lawdom in both worlds. The forces and materials would be different but not the consistency of the concatenation of events.

Aristotelian logic is incomplete and insufficient. It treats only the most simple relations and does not cover the more complicated cases of thinking, but so far as it goes it is without fault. If we grant that all men are mortal and that Caius is a man, we must make the conclusion that Caius is mortal—otherwise he would not be a man but some immortal being, and this would upset the principle of consistency.

We might assume that there are no uniformities in nature, or that all rules have exceptions, or that the uniformities are mere approximations, in which case we would have a world of haphazard happenings. But that would never upset either *Barbara* or *celarent,* or any other rule of pure logic. The items of actual existence would not be classifiable, but the Aristotelian method would not thereby become wrong.

Some time ago I made the following comment on the nature of logic in the *Primer of Philosophy*, (p. 109):

"Mathematicians with great ingenuity have invented various kinds of mathematics. They have shown that Euclidean geometry is but one actual case among many possible instances. Space might be curved, it might be more than three-dimensional. But no one has yet been bold enough to propound a theory of curved reason.

"And why should there not as well exist a curved logic as a mathematics of curved space? A curved logic would be a very original innovation for which no patent has yet been applied for. What a splendid opportunity to acquire Riemann's fame in the domain of logic!"

Now it happens that my friend, Mr. Francis C. Russell of Chicago, received a letter on sundry topics of modern logic from Mr. Charles S. S. Peirce, known as one of the most prominent logicians, and it contains a most interesting passage which sounds like an answer to this challenge of mine. With the permission of the writer I quote it in this connection:

"Before I took up the general study of relatives, I made some investigation into the consequences of supposing the laws of logic to be different from what they are. It was a sort of non-Aristotelian logic, in the sense in which we speak of non-Euclidean geometry. Some of the developments were somewhat interesting, but not sufficiently so to induce me to publish them. The general idea was, of course, obvious to anybody of sufficient grasp of logical analysis to see that logic reposes upon certain positive facts, and is not mere formalism. Another writer afterward suggested such a false logic, as if it were the wildest lunacy, instead of being a plain and natural hypothesis worth looking into [notwithstanding its falsity]."*

I begin to think that Mr. Charles S. S. Peirce understands something else by Aristotelian logic than I do.

* In giving his consent to publish this extract from his letter, Mr. Charles S. S. Peirce sends an additional explanation which is published on page 158 of the present number.

The world has seen many new inventions. Over the telephone we can talk at almost unlimited distances, and some of our contemporaries fly like birds through the air. Radium has been discovered which is often assumed with a certain show of plausibility to upset the laws of physics, but the invention of non-Aristotelian logic would cap the climax. We make bold to prophesy that the non-Aristotelian logic will abolish Aristotle as little as the non-Euclideans have antiquated Euclid. If it comes it will, if it be sound, give us new viewpoints, but it will not abolish one iota of the well-established truths of the old logic. Of course, a non-Aristotelian logic would be "worth looking into," even if it were a vain attempt. *Nous verrons.*

PROFESSOR BERTRAND RUSSELL'S VIEWS.

Since the publication of my two books on this subject, *Kant's Prolegomena* and *The Foundations of Mathematics,* I came across an article by one of the most famous mathematicians of our time, Professor Bertrand Russell of Cambridge, England, a scholar of great erudition and author of many valuable books, among which is an excellent book on *The Foundations of Geometry.* If Professor James or his pragmatist adherents speak of Euclid as superseded and no longer true, they are not to be taken seriously, and there is no need of refuting them; but the case is different when mathematicians of standing make similar declarations. Professor Russell's article on "Recent Work on the Principles of Mathematics," published in the *International Monthly,* is bewildering to me. The very style and presentation of the subject is fascinating, perhaps because the arguments seem paradoxical. At any rate the author's prominence has caused me to reconsider my own position, but I can only say that in spite of his unquestioned authority I cling to my own views. All I can do is to contrast his ideas with my own, and for the sake of fairness I will

quote extensively from his essay so as to let him present his views in his own words.

I shall begin with a quotation which I heartily endorse. Professor Russell says (pp. 84-85):

"Logic, broadly speaking, is distinguished by the fact that its propositions can be put into a form in which they apply to anything whatever. All pure mathematics—arithmetic, analysis, and geometry—is built up by combinations of the primitive ideas of logic, and its propositions are deduced from general axioms of logic, such as the syllogism and the other rules of inference. And this is no longer a dream or an aspiration. On the contrary, over the greater and more difficult part of the domain of mathematics, it has been already accomplished; in the few remaining cases, there is no special difficulty, and it is now being rapidly achieved. Philosophers have disputed for ages whether such deduction was possible; mathematicians have sat down and made the deduction. For the philosophers there is now nothing left but graceful acknowledgements."

My mode of thinking has complied with the demand. I would replace the expression "axioms of logic" by "the principle of consistency," but otherwise I would feel in perfect agreement with Professor Russell, if his article did not abound in many other statements which appear to me irreconcilable with this unequivocal and simple description of the situation.

The reader will notice that Professor Russell is rather hard on philosophers, but it can not be denied that philosophers, at least many men who have gained fame under that name, have unduly slighted mathematics. It is strange, though perhaps natural, that mathematicians like Schroeder and Peano have distinguished themselves in the construction of an algebra of logic. Furthermore there are a number of modern mathematicians, inspired by the broader and more philosophical conceptions of mathematical notions, who have advanced their science by taking new view-points. Professor Russell mentions three great Germans, Weierstrass, Dedekind and Cantor, whose mer-

UMI

its are indubitable. Other names might have been added, such as Clebsch, Grassmann, Fuchs, Klein, Lindemann and Staudt, but I fail to see that any one of them has tried to solve or claims to have solved the philosophical problem of the foundation of mathematics. The great drift of their labors, so far as I can judge, is, with the exception of the work of Grassmann, purely mathematical.

I grant that Euclid has his faults, but I believe that his mistakes can be remedied. I also grant that "he is not an easy author and terribly long winded." I deem his proofs tiresome with the monotonous refrain, *Q. E. D.*, and I would replace his method as suggested above by changing the doctrinary style of propositions into the accomplishment of tasks. But for all that, Euclidean geometry remains classical, and I can not understand Professor Russell's harsh verdict when he says (p. 100):

> "It is nothing less than a scandal that he should still be taught to boys in England. A book should have either intelligibility or correctness; to combine the two is impossible, but to lack both is to be unworthy of such a place as Euclid has occupied in education."

I do not agree with Professor Russell that "to combine the two," (viz., intelligibility and correctness) is "impossible." If that were so we would land in mysticism.

Here is another passage on Euclid. Professor Russell says (p. 98):

> "It has gradually appeared, by the increase of non-Euclidean systems, that geometry throws no more light upon the nature of space than arithmetic throws upon the population of the United States."

True, the formal sciences never supply us with facts; but they offer us a method of dealing with facts, and that is better. Professor Russell continues:

> "Geometry is a whole collection of deductive sciences based on a corresponding collection of sets of axioms. One set of axioms is Euclid's; other equally good sets of axioms lead to other results.

Whether Euclid's axioms are true, is a question as to which the pure mathematician is indifferent; and what is more, it is a question which it is theoretically impossible to answer with certainty in the affirmative. It might possibly be shown, by very careful measurements, that Euclid's axioms are false; but no measurements could ever assure us (owing to the errors of observation) that they are exactly true. Thus the geometer leaves to the man of science to decide, as best he may, what axioms are most nearly true in the actual world."

Since Euclid's geometry consists of constructions of pure thought, since there are no points, lines, surfaces, planes, etc., in the objective world, it is obviously impossible to test the truth of Euclidean propositions by actual measurement. Professor Russell does not define his conception of truth. We would say that a Euclidean proposition is true when it is an adequate or correct description of the results of a construction. The question is not, what axioms are most nearly true in the actual world, but which geometry is most serviceable in calculating the relations that obtain in the actual world.

Professor Russell frequently indulges in mystifications. He says (p. 84):

"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true. People who have been puzzled by the beginnings of mathematics will, I hope, find comfort in this definition, and will probably agree that it is accurate."

All this is *ingeniosius quam verius*. Statements can easily assume a paradoxical form when they are based upon an inaccuracy of terms. Mathematical propositions do not describe realities, but, because lines and planes are not real, we can not say that what mathematics teaches is "not true." Nor is it fair to define mathematics as "the subject in which we never know what we are talking about."

I understand that Professor Russell bases his view upon the method of some Italian mathematicians who avoid

a discussion of the foundation of mathematics by the use of a conditional "if." They start their proposition by saying, "If I do this, the result will be such and such." The "if" sentence is purely hypothetical and they do not trouble about it, but if it be allowed to stand the result can not be denied. Professor Russell explains the situation thus:

"Pure mathematics consists entirely of asseverations to the effect that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true and not to mention what the anything is, of which it is supposed to be true. Both these points would belong to applied mathematics. We start, in pure mathematics, from certain rules of inference, by which we can infer that *if* one proposition is true, then so is some other proposition. These rules of inference constitute the principles of formal logic. We then take any hypothesis that seems assuring, and deduce its consequences. *If* our hypothesis is about *anything*, and not about some one or more particular things, then our deductions constitute mathematics."

We may grant that "the rules of inference constitute the principles of formal logic." But why should it be "essential not to discuss whether the first proposition is really true?" I propose to avoid the vicious "if" which leaves the entire science of mathematics in the air, and to dig down to the bottom rock of our mode of thought and build the foundation that is needed for the superstructure of this noblest and loftiest of all the sciences.

If my conception of mathematics is true we do not need in geometry "a certain number of primitive ideas, supposed incapable of definition and a certain number of primitive propositions or axioms, supposed to be incapable of proof."[4] We remove every trace of particularity and build upon the abstract idea of "anyness" a universe of pure thought which will serve as a model for any possible formation, fictitious or real.

[4] *Ibid.*, p. 84.

Professor Russell lays much stress on the symbolic nature of modern logic, and I grant that the significance of symbolism can not be overrated. I would insist that language of any kind, yea even sense-perceptions, are symbolic, and the very nature of thought is symbolism. Sense-impressions change into sensations and sensations become perceptions solely through becoming symbolic. As soon as a sense-impression of a definite kind has come to represent some fact, an event or an object that causes it, then the sense-perception stands for or symbolizes the fact sensed. This is the origin of thought, and we have defined the soul as "a system of sentient symbols." Professor Russell apparently uses the term "symbol" in the more limited sense of an algebraic symbol. He says (p. 85):

"People have discovered how to make reasoning symbolic, as it is in Algebra, so that deductions are effected by mathematical rules."

Algebraic symbols have the great advantage over language that they are definite and rigid. Language suffers from the fault of being vague. The use of our speech is incredibly loose and even the most common words, such as "to be," "to have," "we," "you," etc. possess several shades of meaning. This is not so in algebra and so logicians hope to overcome the looseness of reasoning in language by the employment of symbols which are as rigidly defined as the algebraic terms. The invention of such terms and of their mode of operation is a difficult task, and it would require a good deal of concentration of thought for any one to familiarize himself with a system of such an algebra of logic; but the gain is rich when we consider that thought acquires thereby the virtue of mathematical exactness. The trouble so far has been that there has been too little cooperation among logicians and almost every one of them invents symbols of his own.

Professor Russell's love of paradox appears in his expo-

sition of the importance of symbolism. He says (pp. 85-86):

"It is not easy for the lay mind to realize the importance of symbolism in discussing the foundations of mathematics, and the explanation may perhaps seem strangely paradoxical. The fact is that symbolism is useful because it makes things difficult. (This is not true of the advanced parts of mathematics, but only of the beginnings.) What we wish to know is, what can be deduced from what. Now, in the beginnings, everything is self-evident; and it is very hard to see whether one self-evident proposition follows from another or not. Obviousness is always the enemy of correctness. Hence we invent some new and difficult symbolism, in which nothing seems obvious. Then we set up certain rules for operating on the symbols, and the whole thing becomes mechanical."

We would not say that "symbolism is useful because it makes things difficult," but because it makes thought exact, and further, though it will prove difficult in the beginning, it will make exact thinking easy. It will show in a formula the machinery of thought and thus will render the process of thinking intelligible. In the same way a beginner in algebra may deem this mode of computation hard, but as soon as he has mastered its principles he will be enabled thereby to solve difficult problems without great exertion.

One of Professor Russell's observations is very good, though again stated in such a way as to make its truth appear in a paradoxical light. We must be on our guard against statements that appeal to us as obvious. The records of the history of philosophy and of religious dogma contain many flagrant instances of ideas deemed to be innate and of truths supposedly so obvious that it was claimed they did not stand in need of any proof. Professor Russell says (p. 86):

"The proof of self-evident propositions may seem, to the uninitiated, a somewhat frivolous occupation. To this we might reply that it is often by no means self-evident that one obvious proposi-

tion follows from another obvious proposition; so that we are really discovering new truths when we prove what is evident by a method which is not evident. But a more interesting retort is, that since people have tried to prove obvious propositions, they have found that many of them are false. Self-evidence is often a mere will-o'-the-wisp, which is sure to lead us astray if we take it as our guide."

When Professor Russell speaks of " a method which is not self-evident" I understand him to mean a method which must first prove its right of existence.

The mathematician should banish from his science any proposition which can show no other title than the claim of self-evidence. For this reason I have endeavored to do away with axioms and to build up mathematics without resorting to assumptions, self-evident statements, or asseverations of any kind. I wish Professor Russell would not describe mathematics as consisting of "asseverations"; the very idea is jarring on my conception of the nature of mathematics.

Among modern mathematicians Professor Peano has distinguished himself by an application of the algebraic method to mathematics in general, and Professor Russell looks up to him as a leader. He says (pp. 86-87):

"The great master of the art of formal reasoning, among the men of our day, is an Italian, Professor Peano, of the University of Turin. He has reduced the greater part of mathematics (and he or his followers will, in time, have reduced the whole) to strict symbolic form, in which there are no words at all. In the ordinary mathematical book, there are no doubt fewer words than most readers would wish. Still, little phrases occur, such as *therefore, let us assume, consider,* or *hence it follows.* All these, however, are a concession, and are swept away by Professor Peano. For instance, if we wish to learn the whole of arithmetic, algebra, the calculus, and indeed all that is usually called pure mathematics (except geometry), we must start with a dictionary of three words. One symbol stands for *zero,* another for *number,* and a third for *next after.* What these ideas mean, it is necessary to know if you

wish to become an arithmetician. But after symbols have been invented for these three ideas, not another word is required in the whole development. All future symbols are symbolically explained by means of these three. Even these three can be explained by means of the notions of *relation* and *class*; but this requires the logic of relations, which Professor Peano has never taken up."

Further down on page 99 Professor Russell says:

"One great advance, from the point of view of correctness, has been made by introducing points as they are required, and not starting, as was formerly done, by assuming the whole of space. This method is due partly to Peano, partly to another Italian named Fano. To those unaccustomed to it, it has an air of somewhat wilful pedantry. In this way, we begin with the following axioms: (1) There is a class of entities called *points*. (2) There is at least one point. (3) If *a* be a point, there is at least one other point besides *a*. Then we bring in the straight line joining two points, and begin again with (4) namely, on the straight line joining *a* and *b*, there is at least one other point besides *a* and *b*. (5) There is at least one point not on the line *ab*. And so we go on, till we have the means of obtaining as many points as we require. But the word *space*, as Peano humorously remarks, is one for which geometry has no use at all."

There is no need of using the word "space," but is not the idea of space of some kind presupposed in the notion of a line, or even in the notion of a point? What is a point except a spot in space? Professor Russell must excuse me for finding Professor Fano's method of avoiding the difficulty comical. He starts "There is a class of entities called *points*. There is at least one point. If *a* be a point, there is at least one other point besides *a*." This is all very nice and begins like a fairy-tale, "Once upon a time." He rushes these statements upon us with an unmitigated abruptness which is truly naive. He has points, lines, distances, directions, but knows nothing of space. The very word "space" is abolished! Such are "the rigid methods employed by modern geometers" that "have deposed Euclid from his pinnacle of correctness"!

I feel strongly inclined to enter into Professor Russell's discussion of Zeno's problem, but space forbids. It would take an essay by itself, but a few comments on the subject may be permitted. Professor Russell presents the issues so interestingly that I wish I could read the whole exposition to my readers. A sample will prove that this is not mere courtesy. Professor Russell speaks of the infinitesimal as follows (pp. 89-90):

"The infinitesimal played formerly a great part in mathematics. It was introduced by the Greeks, who regarded a circle as differing infinitesimally from a polygon with a very large number of very small equal sides. It gradually grew in importance, until, when Leibnitz invented the infinitesimal calculus, it seemed to become the fundamental notion of all higher mathematics. Carlyle tells, in his *Frederick the Great*, how Leibnitz used to discourse to Queen Sophia Charlotte of Prussia concerning the infinitely little, and how she would reply that on that subject she needed no instruction—the behavior of courtiers had made her thoroughly familiar with it. But philosophers and mathematicians—who for the most part had less acquaintance with courts—continued to discuss this topic, though without making any advance. The calculus required continuity, and continuity was supposed to require the infinitely little; but nobody could discover what the infinitely little might be. It was plainly not quite zero, because a sufficiently large number of infinitesimals, added together, were seen to make up a finite whole. But nobody could point out any fraction which was not zero, and yet not finite. Thus there was a deadlock."

So far as I know, mathematicians have never taken this deadlock seriously, for they know that the infinitesimal is a fiction. There are no infinitesimals in the objective world, and in the ideal realm of mathematics it is an attempt to represent a continuum under the aspect of discrete units, which is necessary for the purpose of computation.

We have stated above that all thought is symbolic, and the method of thought depends upon the symbols we employ. There are two possibilities; we can proceed either in a path of uninterrupted motion or we may cover the

ground in steps. The former method is geometrical, the latter arithmetical. The former is a continuous progress, the latter an advance in counting units. The former has the advantage of presenting outlines of pictures in their totality as images; it is qualitative. The latter sums up numbers fit for use in computations; it is quantitative. Now it so happens that sometimes we need one and sometimes the other. A geometrical curve is a continuum, and so if we wish to calculate it we must change it into a series of units with a constant change of direction. The smaller we make these units the more accurate becomes our approximation; only if they could be made zero, would they be correct. But since we needs must conceive them as being ultimately concrete, rectilinear lines, they are treated as infinitesimals. The very idea is an unrealizable fiction, but it serves the purpose of a best possible approximation in describing a continuum in terms of discrete units.

But if the infinitesimal is unreal, because it is a fiction, how can it be useful? We must consider that it is a fiction which serves a purpose. There is a difference between "fiction" and "a fiction." Every mathematical concept is "a fiction" in the sense that it is not a thing, not an actual reality, not a concrete bodily object, but a product of thought, *ein Gedankenwesen,* as Kant calls it. If we treat a product of pure thought as if it were a concrete thing of objective reality we become involved into contradictions and are nonplussed. Here the indefiniteness of language proves a valuable help to mystagogues. We can make paradoxical statements about any mathematical term by an ambiguous use of such words as real, actual, true, etc. We may mean by "real" the concrete materiality of a thing, its definite effciency in existence, or its objective significance. Thus the polar axis around which the earth turns may be called real or absolutely unreal, purely ideal or definite and actual.

In consideration of the paramount significance of relations (i. e. the purely formal aspect of things) the ancient mystic thinker of China said: "Existence makes things actual,[5] but the non-existent in them makes them useful" (*Lao-Tze's Tao Teh King*, Ch. 11).

One of Euclid's postulates declares that "the whole is greater than any of its parts," and we accept this truth for magnitudes; so far as I can see it can have no meaning when applied to items in which the quality of magnitude is absent. Take for instance the purely formal laws of the universe. They are a part of objective reality and yet their sphere of application may truly be said to be larger than that of the whole of which they form a part. By an *a priori* construction they have been developed in the subjectivity of the human mind and their sphere of efficiency applies to any possible world.

The same idea can be stated in religious terms thus: God is part of the All, yet God is greater than the All.

Professor Russell proposes for refutation a maxim shaped in imitation of this same postulate of Euclid. He says when speaking of the evasive nature of obviousness and self-evidence (p. 86):

"For instance, nothing is plainer than that a whole always has more terms than a part, or that a number is increased by adding one to it. But these propositions are now known to be usually false. Most numbers are infinite, and if a number is finite you may add ones to it as long as you like without disturbing it in the least."

Mark the difference. "The whole is greater than any of its parts" and "the whole has always more terms than a part." Can we not describe the same thing in one term and in an infinite series of terms, as for instance:

$$1 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \textit{ad infinitum}.$$

[5] The common translation of *li* here is "profitable," and the etymology of the character which contains the roots "knife" and "harmony" indicates a meaning such as "cutting" or "efficient." The word is now used in the sense of "sharp." We might translate "pragmatic." But in the present passage it stands in contrast to "useful," and so I prefer the reading "actual" or "real."

The number of terms in which we cast our formula is not identical with the thing described. "One" is not infinite even though we can express it in an infinite series. If we ignore the difference between the thing and the terms in which it is expressed and count the terms numerically or quantitatively with an absolute disregard of their qualitative value, we are compelled to accept Zeno's solution of the problem, that Achilles can not overtake the tortoise in a running match. Professor Russell recapitulates this old conundrum thus (pp. 95-96):

"Let Achilles and the tortoise start along a road at the same time, the tortoise (as is only fair) being allowed a handicap. Let Achilles go twice as fast as the tortoise, or ten times or a hundred times as fast. Then he will never reach the tortoise. For at every moment the tortoise is somewhere, and Achilles is somewhere; and neither is ever twice in the same place while the race is going on. Thus the tortoise goes to just as many places as Achilles does, because each is in one place at one moment, and in another at any other moment. But if Achilles were to catch up with the tortoise, the places where the tortoise would have been, would be only part of the places where Achilles would have been. Here, we must suppose, Zeno appealed to the maxim that the whole has more terms than the part. Thus if Achilles were to overtake the tortoise, he would have been in more places than the tortoise; but we saw that he must, in any period, be in exactly as many places as the tortoise. Hence we infer that he can never catch the tortoise. This argument is strictly correct, if we allow the axiom that the whole has more terms than the part. As the conclusion is absurd, the axiom must be rejected, and then all goes well. But there is no good word to be said for the philosophers of the past two thousand years and more, who have all allowed the axiom and denied the conclusion."

While Professor Russell speaks of Zeno's conclusion as "absurd," and therefore rejects it, he regards the paradox of Tristram Shandy as a mere "oddity" which is a "paradoxical but perfectly true proposition." He says (pp. 96-97):

"The retention of this axiom leads to absolute contradictions, while its rejection leads only to oddities. Some of these oddities,

it must be confessed, are very odd. One of them, which I call the paradox of Tristram Shandy, is the converse of the Achilles, and shows that the tortoise, if you give him time, will go just as far as Achilles. Tristram Shandy, as we know, employed two years in chronicling the first two days of his life, and lamented that, at this rate, material would accumulate faster than he could deal with it, so that, as years went by, he would be farther and farther from the end of his history. Now I maintain that, if he had lived forever, and had not wearied of his task, then, even if his life had continued as eventfully as it began, no part of his biography would have remained unwritten. For consider: the hundredth day will be described in the hundredth year, the thousandth in the thousandth year, and so on. Whatever day we may choose as so far on that he cannot hope to reach it, that day will be described in the corresponding year. Thus any day that may be mentioned will be written up sooner or later, and therefore no part of the biography will remain permanently unwritten. This paradoxical but perfectly true proposition depends upon the fact that the number of days in all time is no greater than the number of years."

I hesitate to say that these two series are equal:

1+1+1+1+1 etc., without end, and

365+365+365+365+365 etc., also without end.

Yet if an infinite number of days will cover an infinite number of years the two series ought to be equal. I am afraid we shall all be hurled into infinity before we can find out the truth as to whether an infinity of days is as large as an infinity of years. If they are equal I should like to know what part the difference will play, since it will be the sum of an infinite series of 364 in each term.

I doubt whether the perverted form of Euclid's axiom is to be blamed (as Professor Russell thinks) for the deadlock to which Zeno's fallacy leads. I would say in explanation of the paradox that an infinite series need not be an actual infinitude. An infinite series is a mental operation, while an infinitude is the objective extension without end. An infinite series sometimes describes a very finite magnitude. For instance, 0.333..... is an infinite

decimal fraction, but the infinite series of its terms (0.3+0.03+0.003....) does not involve that it represents an infinitude. It would take an infinitude to write all the decimals out in their completeness, but for that reason its value (say ⅓ of a second) is quickly passed and is not equal to any other infinite series, as for instance a third of an hour.

An infinite series is a function, and the essential feature of a function is the arrangement and not the number of its terms. If their number is limited we can sum up the *facit*; if it is unlimited or infinite we can never finish the function,—we can only approximate it or must let it stand, but the *facit* has nothing to do with it. It follows from this that two infinite series are not necessarily alike, because they are both infinite. They differ according to their terms and the arrangement of their terms. Here I am in full agreement with Professor Russell when he says, "It must not be supposed that all infinite numbers are equal" (p. 95), and rightly insists on the significance of "the way in which the terms are arranged" (p. 94), and the "particular type of order" (p. 97).

Professor Russell seeks the root of the trouble in the infinitesimal, but it lies there only if we forget the vague character of the infinitesimal, and expect it to be a definite magnitude to boot.

If we had to regard infinitesimals as actual and objective existences, there would be no such things as the next moment, and the smallest part. But in order to prove it we must be careful *not* to think of "moment" as a short yet definite measure of time. We could not prove our case if we said there is no such a thing as the next hour, or minute, or second, or jiffy. We must identify (as does Professor Russell) the word "moment" with the term infinitesimal, viz., the smallest possible fraction of time.

The same is true if we divide a piece of matter. We may come down to very small bits but shall never reach an

infinitesimal. "Nevertheless," says Professor Russell, "there *are* points, only they are not reached by successive division" (p. 91).

True, very true! Yet while in my opinion the propositions that "there is no next moment" and "there are no smallest particles" are due to the notion of the infinitesimal if conceived as an actual existence, Professor Russell attributes these very paradoxes to the abolition of the infinitesimal. He says (pp. 90-91):

"But at last Weierstrass discovered that the infinitesimal was not needed at all, and that everything could be accomplished without it. Thus there was no longer any need to suppose that there was such a thing....

"The banishment of the infinitesimal has all sorts of odd consequences, to which one has to become gradually accustomed. For example, there is no such thing as the next moment....

"The same sort of thing happens in space.... we never reach the infinitesimal in this way."

Professor Russell rejects the infinitesimal but accepts the infinite and he defines it, too. He says (pp. 92-93):

"The philosophy of the infinitesimal, as we have just seen, is mainly negative. People used to believe in it, and now they have found out their mistake. The philosophy of the infinite, on the other hand, is wholly positive. It was formerly supposed that infinite numbers, and the mathematical infinite generally, were self-contradictory. But as it was obvious that there were infinities—for example, the number of numbers—the contradictions of infinity seemed unavoidable....

"Twenty years ago, roughly speaking, Dedekind and Cantor asked this question [What is infinity?], and, what is more remarkable, they answered it. They found, that is to say, a perfectly precise definition of an infinite number or an infinite collection of things. This was the first and perhaps the greatest step. It then remained to examine the supposed contradictions in this notion. Here Cantor proceeded in the only proper way. He took pairs of contradictory propositions, in which both sides of the contradiction would be usually regarded as demonstrable, and he strictly examined the supposed proofs. He found that all proofs adverse to infinity in-

volved a certain principle, at first sight obviously true, but destructive, in its consequences, of almost all mathematics. The proofs favorable to infinity, on the other hand, involved no principle that had evil consequences. It thus appeared that common sense had allowed itself to be taken in by a specious maxim, and that, when once this maxim was rejected, all went well.

"The maxim in question is, that if one collection is part of another, the one which is a part has fewer terms than the one of which it is a part. This maxim is true of finite numbers. For example, Englishmen are only some among Europeans, and there are fewer Englishmen than Europeans. But when we come to infinite numbers, this is no longer true. This breakdown of the maxim gives us the precise definition of infinity. A collection of terms is infinite when it contains as parts other collections which have just as many terms as it has. If you can take away some of the terms of a collection, without diminishing the number of terms, then there are an infinite number of terms in the collection."

I am somehow not satisfied with this definition; nor am I more enlightened through the example adduced for the sake of explanation (p. 93):

"For example, there are just as many even numbers as there are numbers altogether, since every number can be doubled. This may be seen by putting odd and even numbers together in one row, and even numbers alone in a row below:

1, 2, 3, 4, 5, *ad infinitum.*
2, 4, 6, 8, 10, *ad infinitum.*

There are obviously just as many numbers in the row below as in the row above, because there is one below for each one above. This property, which was formerly thought to be a contradiction, is now transformed into a harmless definition of infinity, and shows, in the above case, that the number of finite numbers is infinite."

These several views of Professor Russell on the infinitesimal and the infinite do not seem to me quite consistent. But we shall hear from him again. He claims that there is a greatest infinite number while Cantor has offered a proof that there is none. Professor Russell says (p. 95):

"There is a greatest of all infinite numbers, which is the number of things altogether, of every sort and kind. It is obvious that

there cannot be a greater number than this, because, if everything has been taken, there is nothing left to add. Cantor has a proof that there is no greatest number, and if this proof were valid, the contradictions of infinity would reappear in a sublimated form. But in this one point, the master has been guilty of a very subtle fallacy, which I hope to explain in some future work."

I believe that most mathematicians will side with Cantor. We claim that "the number of things altogether of every sort and kind," is not and can never be "the greatest of all infinite numbers." For suppose we would count all things of every sort and kind, and we had accomplished the task, we could add to it one or two or a few thousand units, we could multiply it with itself and so *ad infinitum*.

So far as I understand the nature of number there can no more be a highest number than there can be an end to space and time.

PROFESSOR BERTRAND RUSSELL'S CRITICISM.

Professor Russell's love of paradox renders his article interesting, but while it makes the reading of it pleasant, I am aware that it sometimes obscures the meaning. Having given it a careful and repeated perusal I am not sure that I have always rightly interpreted his humor. His censure of Euclid may be of this kind. We may agree better than it seemed to me at the first reading.

The problems concerning the foundations of geometry and of mathematics in general are by no means so definitely settled that one solution may be said to have acquired the consensus of the competent, and for this reason I feel that a little mutual charity is quite commendable. I have found it wanting mainly in those circles which represent the two extremes, the old-fashioned Euclideans and the new-fangled non-Euclideans; they scorn and condemn all who look at the problem through some other spectacles than their own. But I am glad to notice that Professor

Russell is not one of these. He can review considerately and kindly the work of one who differs from him on a subject to which he himself has given a great deal of attention. Therefore I here express publicly my recognition of the gentlemanly tone of Professor Russell's review, and having monopolized the floor myself in criticising him, I deem it but just to let him have his turn.

In the *Mathematical Gazette,* Vol. V, No. 80 (June-July, 1909), pp. 103-104, Professor Russell, speaking of my recent work, *The Foundations of Mathematics,* says:

"This book is a more or less popular exposition of a philosophy of geometry which is, in its main outlines, derived from Kant. The main title, if uncorrected by the sub-title, would be somewhat misleading, since the foundations of arithmetic and analysis are not discussed, but only the foundations of geometry. The author begins by a brief account of the development of non-Euclidean geometry, which is followed by much longer chapters "on the philosophical basis of mathematics" and on "mathematics and metageometry." The historical chapter, though it does not profess to give more than a sketch, might with advantage have been enlarged by some account of projective geometry and the projective treatment of metrics. Dr. Carus speaks always as though non-Euclidean straight lines were not really straight, but were merely called straight out of wilfulness. The projective treatment shows, better than the metrical, wherein the straight lines of non-Euclidean spaces agree with those of Euclid, and ought therefore not to be omitted even in a mere outline. It would seem also that Dr. Carus regards a three-dimensional non-Euclidean space as necessarily contained in a four-dimensional Euclidean space, for he asks "what Riemann would call that something which lies outside of his spherical space," apparently not realizing that spherical space does not require anything outside it.

"The author's philosophical theory of geometry may be summarized as follows. Geometry, like logic and arithmetic, is *a priori* but it is not *a priori* in the same degree as logic and arithmetic. There is the *a priori* of being and the *a priori* of doing, and geometry belongs to the latter: it is derived from the contemplation of motion, and can be constructed from the 'principles of reasoning and the privilege of moving about.' We know *a priori* what are the possibilities of motion; thus, although there is nothing *logically* impossible about the

assumption of four dimensions, yet 'as soon as we make an *a priori* construction of the scope of our mobility, we find out the incompatibility of the whole scheme.' The *a priori* is identical with the purely formal, which originates in our minds by abstraction; it is applicable to the objective world because the materials of formal thought are abstracted from the objective world.

"Most of the arguments in the book lead one to expect that Euclid will be declared to be certainly alone valid as against non-Euclidean geometry, yet this is not the conclusion drawn by the author. He says: 'The result of our investigation is quite conservative. It re-establishes the apriority of mathematical space, yet in doing so it justifies the method of metaphysicians in their constructions of the several non-Euclidean systems.... The question is not, "Is real space that of Euclid or of Riemann, of Lobatchevsky or Bolyai?" for real space is simply the juxtaposition of things, while our geometries are ideal schemes, mental constructions of models for space measurement. The real question is, "Which system is the most convenient to determine the juxtaposition of things?"' (p. 121). Yet a few pages later he says: 'The theorem of parallels is only a side issue of the implications of the straight line' (p. 129). It is not clear how these statements are reconciled, for the earlier statement seems to imply that there is no 'theorem' of parallels at all.

"A few of the author's assertions are somewhat misleading. For example, he states, as a fact not open to controversy, that Euclid's axiom or postulate of parallels originally occurred first in the proof of the twenty-ninth proposition, not being mentioned either among the axioms or among the postulates (p. 2). On the other hand, Stäckel and Engel (*Theorie der Parallellinien*, p. 4) say that, following Heiberg, they do not regard the postulate of parallels as a later addition, which would seem to show that Dr. Carus's opinion is at least open to question. Again he says (p. 84): "While in spherical space several shortest lines are possible, in pseudospherical space we can draw one shortest line only.' As regards spherical space, the more exact statement is that in general only one shortest line can be drawn between two given points, but when the two points are antipodes, an infinite number of shortest lines can be drawn between them.

"The book concludes with an epilogue, in which the existence and attributes of the Deity are deduced from the nature of mathematical truth."

With reference to Professor Russell's several comments I will make these statements:

(1) The title of my book read originally "The Foundation of Geometry," but since this designation had been forestalled by Professor Hilbert's book I changed it to "The Foundations of Mathematics" with the subtitle "A Contribution to the Philosophy of Geometry," to make up by it for what may be misleading in the main title.

(2) Though I will grant that a discussion of projective geometry might be added to advantage in an exposition of non-Euclidean geometry, I doubt whether it will help us much in laying the foundation of geometry. I am inclined to think that it might complicate the problem and confound the issue.

(3) I am indeed of the opinion that the use of the term "straight line" had better be limited to the straight line of Euclidean space and that its analogies in other spaces should be named "straightest lines" or be designated by any other term that might be deemed appropriate.

(4) I conceive every kind of space conception as independent and grant that none of them ought to be thought of as being constructed in Euclidean space. But if space is a scope of motion, I can not think of a space that is limited. Spherical space ought to be conceived as possessed of a spherical drift, but for that it ought to be infinite. If it is not infinite, I would ask the question, what is outside? In my opinion we can not get rid of infinitude. The straightest lines in spherical space would not be infinite. They would be merely boundless. Outside of every boundless spherical line we must be able to construct other lines or spherical surfaces and thus spherical space would be as infinite as Euclidean space. I may be wrong but I am willing to learn.

(5) The passage on page 84 is an obvious mistake. When I wrote it I had in mind the Mercator projection

of the globe where both the meridians and the parallels assume the same straightness as the straight lines of Euclidean space. The parallels (so called by geographers) are not shortest lines, but if the parallels on the globe, because they represent straight lines in the Mercator projection, were called straightest lines, we could make them enclose a space with shortest lines on the sphere, or with the parallels of another equatorial system. Of course not being truly shortest lines, the statement is a mistake and I am much obliged to Professor Russell for having called my attention to it.

PARALLELISM AND INFINITY. A COMMENT ON MR. FRANCIS C. RUSSELL'S THEOREM.

Mr. Francis C. Russell, an American namesake of Professor Bertrand Russell, of Cambridge, England, stands up so doughtily for Euclid that he has excited the wrath of non-Euclideans. In his retort courteous to the strictures of his critics he proposes a theorem which would upset Lobatchevsky and Bolyai, if the right angle were and remained the same in non-Euclidean space as in plane geometry. But this is exactly the crux. We need more rigidity in the use of terms.

Our geometricians, Euclid as well as the non-Euclideans, have not always defined with sufficient precision all the names and notions which they introduce. A right angle according to my conception of geometrical notions belongs to the important class of boundary conceptions which on account of their uniqueness become standards of measurement.

The straight line (corresponding to a crease in a sheet of paper folded upon itself) halves the plane, while the right angle (corresponding to the folded sheet of paper again folded upon itself) represents the halved half of the plane. There are innumerable curves, but only one

straight line; and there are innumerable acute and innumerable obtuse angles, but only one right angle. Accordingly we may define the right angle as the angle which represents one-quarter of the entire sweep of direction round a common center. The right angle on a sphere is smaller than in plane geometry; it increases and decreases with the length of the radius and becomes approximately equal to the plane right angle only when the radius becomes infinitely great.

Mr. Russell proves in his theorem that the right angle remains a right angle, but the Russell theorem holds good only for Euclidean space. The right angles in other spaces follow the law of their spaces.

In the same way the term parallel has another sense in plane than in projective geometry. We are told that "lines which meet at infinity are called parallel," but if two lines are truly parallel will they not remain parallel even in infinity? We may freely grant that they will meet at infinity, but it would be better not to introduce this feature in the definition.

The cause of much trouble must be sought in the use of Euclidean terms in a non-Euclidean sense. By a straight line mathematicians formerly understood the straight line in the Euclidean plane, and if we now become acquainted with other lines called straight because they somehow correspond to the straight line of the Euclidean plane, the layman who is only superficially acquainted with the new geometry is naturally puzzled. It is therefore advisable to call the non-Euclidean straight line a "straightest line," or to give it some other suitable name so as to distinguish it from the straight line in the plane.

The same is true of parallels. So far as my linguistic feeling is concerned I cannot overcome the original meaning of the etymology of parallel. Parallel lines are to me lines which remain the same distance apart; thus rail-

road tracks, whether straight or curved, are parallel; and since this running side by side is the original sense of the word, they deserve to be so called. The geographer too uses the word parallel in its etymological meaning. If straight lines are parallel (i. e., keep at the same distance) they do not meet even in infinity,—although I will grant anything for infinity.

Infinity is the land of mathematical hocus pocus. There Zero the magician is king. When Zero divides any number he changes it without regard to its magnitude into the infinitely small; and inversely, when divided by any number he begets the infinitely great. In this domain the circumference of the circle becomes a straight line, and then the circle can be squared. Here all ranks are abolished, for Zero reduces everything to the same level one way or another. Happy is the kingdom where Zero rules!

I do not say that the notion of infinity should be banished; I only call attention to its exceptional nature, and this so far as I can see, is due to the part which zero plays in it, and we must never forget that like the irrational it represents a function which possesses a definite character but can never be executed to the finish. If we bear in mind the imaginary nature of these functions, their oddities will not disturb us, but if we misunderstand their origin and significance we are confronted by impossibilities.

Since in infinity all is reduced to a democratic sameness the points on two parallel lines coincide when they lie at infinity. And the same is true of planes. Suppose we lay down a plane on the absolutely smooth surface of the ocean. It would be tangential to the earth, and the entire range of points at infinity would lie on a line which we will call its infinity-horizon. It is the circumference of a circle, all the diameters of which are infinite straight lines. Now consider that just as two parallel lines cut each

other at infinity, so two parallel planes cut each other in a line at infinity, and we must accept the conclusion that all the infinite number of parallel planes of the entire system of all possible parallel planes upward to an infinite distance and downward to an infinite distance meet at an infinite distance in one line. All their infinity-horizons coincide. Another such a system of planes set at right angles on the horizontal level would behave in the same way. All the perpendicular planes meet in a line which we may call their infinity- meridian. The infinity-meridian is a line which at infinity passes through all the horizontal planes at the point where it cuts their infinity-horizon.

Here the point at infinity is a line and the line is a point.

The infinity-meridian of all the infinite number of perpendicular planes cuts the infinity-horizon of every single plane of the infinite number of horizontal planes, and there are no points in the infinity-meridian which do not cut some of the horizontal planes at infinity. But since all the horizontal planes have only one infinity-horizon, the range of points of the infinity-meridian all lie in the very same point of the infinity-horizon. In other words, the infinity-meridian is one point on the infinity-horizon, and with this point the entire range of its other points will be found to coincide. Thus this line (the infinity-meridian) shrinks into one point and nothing of the infinite extent of the line lies outside this point. Inversely the infinity-horizon is a mere point on the infinity-meridian.

The mathematician may well turn mystic when he moves in infinitudes.

KANTISM AND SPACE.

In a recent number of *The Mathematics Teacher,* Prof. E. D. Roe, Jr., publishes an article under the title "Some Thoughts on Space." He is a Kantian who maintains that "it does not seem that Kant's fundamental principle

that time and space in general are necessary forms of intuition is overthrown." He says (p. 35):

"The simplest hypothesis is not that a handing over to a blank, which is unintelligible, took place, but that reason by its own spontaneity, at least, acted according to the laws of its own activity, and applied its forms and categories to the matter of experience..... If the mind is not constituted so to act space will not result no matter how much experience is had. A cow has the same empirical occasions, but does any one imagine that mathematical space is in a cow's mind, or ever will be, or that it could be revealed to or handed over to the cow, or by any instruction be conveyed to or gotten into the cow's consciousness? If experience could cause its genesis it would be there, for a cow has as much experience as we, doubtless more, because the cow has nothing but experience and all of it in time and space."

Professor Roe differs from Kant in one point. He says (p. 36):

"Kant denied their objective reality. But he should not have done so as by his theory he did not know what was external to the mind. He should have neither denied nor asserted this."

I will not enter here into a discussion of either Kant's or Mr. Roe's doctrine of space. I will only quote what Mr. Roe says of my position:

"The preceding conclusion was reached independently. My attention has since been called to the book of Dr. Carus, *The Foundations of Geometry,* Chicago, The Open Court Publishing Co., 1908. It is a great pleasure to recognize in him a friend and not an antagonist. The results here reached seem to agree with those of Dr. Carus, though the method and standpoint are a little different."

What Kant calls "the empirical occasions" are the single and innumerable experiences we have in life. They are the facts from which we derive our general idea of motion. This general idea of motion is an abstraction. It is not real motion, but the thought of motion. All consideration of energy is omitted from it, it is what Kant would call "pure motion," and the *a priori* constructions

of it, such as lines, angles, triangles, etc., have nothing to do directly with any empirical occasions. The cow can not think in abstractions, and this is the reason why a cow knows nothing of mathematical space.

Mr. Roe objects to the term "abstraction," because it "conveys the idea of unreality." First I would answer that there is no harm in this, for the whole of mathematics is an ideal construction, and it prospers well in the atmosphere of unreality—unreality in the sense that it does not consist of concrete material objects. But secondly I would add that abstractions describe features of real things, and though abstractions as such are not concrete they represent qualities which are real.

Mr. Roe introduces the word "spontaneity" to serve in the place of abstraction, but we are little helped thereby. Spontaneity means self-motion, and it is to be feared that it will involve us into questions as to the nature of mental activity. It is certainly not a simpler conception than pure motion; and actual motion of some kind is absolutely needed in order to give us the abstract notion of either motion or spontaneity. Professor Roe suggests:

> "Why might not one lie perfectly still with eyes closed and receive tactile sensations on different parts of his body and some notion of here and there be called out without the necessity of subjective motion?"

If the notions of "here and there" could originate under these conditions they would be the product of referring to tactile impressions "here" and other impressions, "there." A normal man would use his hand to localize sensations. We do not know which tooth hurts us until we touch the sore spot, but suppose the localization were roughly done internally, we could accomplish it only by allowing our attention to move about from the place of one sensation to that of another. Motion is indispensable for any space-construction.

UMI

I will abstain from discussing other points and will only say that such words as "faculty," which Mr. Roe introduces, had better be avoided.

* * *

Professor Bertrand Russell makes the statement that I have derived my philosophy of geometry in its main outlines from Kant, but that is true in a certain sense only. I have started from Kant and retain much of his terminology, but in the most essential points I have come to conclusions diametrically opposed to his.[6] I might as well consider myself a disciple of Grassmann, although I did not become acquainted with his extension theory until later in life; but he was my teacher in mathematics, and I may unconsciously have imbibed from him many notions which like fertile seeds grew up in my mind and are now inextricably intertwined with my own thoughts.

The statement is often made that our mathematical conceptions depend upon our senses. If we had different eyes or other organs of sense, it is claimed, we would have a different notion concerning space. But this is true only so far as our physiological space-conception is concerned, and even there the modification would be slight. It is difficult to prognosticate what space-notions we would have if we were endowed with an electric sense, but it stands to reason that even the perception of electric shocks or the ability to perceive a discharge of electric shocks upon our surroundings would change nothing in our notion of space, for it would have to be interpreted ultimately with the assistance of the sense of touch, which is and will remain the foundation of all sense-perception. I am convinced that the ability to move with great rapidity, which would be acquired by the faculty of flight, would modify our space-conception more than the possession of electric or

[6] For details see my discussion of Kant's doctrines in my little book *Kant's Prolegomena*.

any other additional sense-organs. After all, our notion of space is ultimately based on the self-observation of our own motion. Without motion no space-conception.

Physiological space is dependent to a great extent on our physiological constitution, and the latter again depends upon the conditions in which we live. The feelings "upward" and "downward" are decidedly different, being subject to the factor of gravitation; and mathematically congruent figures in different positions appear different to us on account of the distinction we make between upward and downward, high and low, right and left.[6]

Mathematical space differs from physiological space in being of a more abstract nature. For its construction we need the idea of pure motion alone, which is treated according to the rules of consistency, analogous to pure logic. Accordingly pure mathematics does not depend upon the senses but is the product of the mind. If rational beings, differing from ourselves, have developed on other planets, they might have different notions of physiological space than we have, but they would have the same logic, the same arithmetic, the same geometry, and all the complications derived therefrom.

Helmholtz once pointed out that rational beings who were moving on a sphere would develop a spherical geometry; but, strange to say, he at the same time overlooked the fact that man is actually living on a sphere, and yet our geometry, as it developed in history, is plane and not spherical. We shall scarcely be mistaken when we say that were we to visit other planets we should nowhere find a race of rational beings who developed either the geometry of Lobatchevsky, or Bolyai, or any other non-Euclidean system, before they would discover that plane geometry was also possible. The non-Euclidean geometries prove

[6] For details see Mach's investigations on "Physiological as Distinguished from Geometrical Space," in his excellent little book *Space and Geometry* (Chicago: Open Court Pub. Co., 1906).

that mathematical concepts of any kind are *a priori* in Kant's sense, which means that they are purely mental constructions. The invention of non-Euclidean systems is not useless, for they enable us to generalize our space-conception and subsume geometrical propositions under larger conceptions of different or higher manifoldnesses. We are not able to visualize some of the non-Euclidean spaces, which means we cannot form definite sense-perceptions of them; but we can think them and establish their several laws in abstract formulas. This generalization is a gain because it enables us better to understand the nature of mathematics in general as well as in its particular propositions. However, metageometricians go too far and misunderstand the significance of non-Euclidean geometries, if they treat mathematical space-conceptions as actualities and expect the rival claims of Euclidean and non-Euclidean systems to be decided before the tribunal of the *a posteriori,* i. e., of experience.

CONCLUSION.

Having laid the foundation of geometry without resorting to axioms, merely through the function of pure motion, the latter being ultimately derived from experience through abstraction, by omitting everything particular, I feel confident that I have furnished a conception which satisfies all demands and will be serviceable for all practical purposes. I avoid the mysticism which necessarily results from other interpretations. I may have overlooked applications of importance but I feel confident that all difficulties can be overcome, and that in the main my solution is on the right track.

Dixi et salvavi animam meam.

EDITOR.

THE FUTURE OF MATHEMATICS.[1]

TO foresee the future of mathematics, the true method is to study its history and its present state.

Is this not for us mathematicians in a way a professional procedure? We are accustomed to *extrapolate,* which is a means of deducing the future from the past and present, and as we well know what this amounts to, we run no risk of deceiving ourselves about the range of the results it gives us.

We have had hitherto prophets of evil. They blithely reiterate that all problems capable of solution have already been solved, and that nothing is left but gleaning. Happily the case of the past reassures us. Often it was thought all problems were solved, or at least an inventory was made of all admitting solution. And then the sense of the word solution enlarged, the insoluble problems became the most interesting of all, and others unforeseen presented themselves. For the Greeks a good solution was one employing only ruler and compasses; then it became one obtained by the extraction of roots, then one using only algebraic or logarithmic functions. The pessimists thus found themselves always outflanked, always forced to retreat, so that at present I think there are no more.

My intention therefore is not to combat them, as they are dead; we well know that mathematics will continue to develop, but the question is how, in what direction?

[1] Translated from the French by George Bruce Halsted.

UMI

You will answer, "in every direction," and that is partly true; but if it were wholly true it would be a little appalling. Our riches would soon become encumbering, and their accumulation would produce a medley as impenetrable as the unknown truth was for the ignorant.

The historian, the physicist even, must make a choice among facts; the head of the scientist, which is only a corner of the universe, could never contain the universe entire; so that among the innumerable facts nature offers, some will be passed by, others retained.

Just so, *a fortiori,* in mathematics; no more can the geometer hold fast pell-mell all the facts presenting themselves to him; all the more because he it is, almost I had said his caprice, that creates these facts. He constructs a wholly new combination by putting together its elements; nature does not in general give it to him ready made.

Doubtless it sometimes happens that the mathematician undertakes a problem to satisfy a need in physics; that the physicist or engineer asks him to calculate a number for a certain application. Shall it be said that we geometers should limit ourselves to awaiting orders, and, in place of cultivating our science for our own delectation, try only to accommodate ourselves to the wants of our patrons? If mathematics has no other object besides aiding those who study nature, it is from these we should await orders. Is this way of looking at it legitimate? Certainly not; if we had not cultivated the exact sciences for themselves, we would not have created mathematics the instrument, and the day the call came from the physicist, we would have been helpless.

Nor do the physicists wait to study a phenomenon until some urgent need of material life has made it a necessity for them; and they are right. If the scientists of the eighteenth century had neglected electricity as being in their eyes only a curiosity without practical interest, we

should have had in the twentieth century neither telegraphy, nor electro-chemistry, nor electro-technics. The physicists, compelled to choose, are therefore not guided in their choice solely by utility. How then do they choose between the facts of nature? We have explained it in a previous article:[2] the facts which interest them are those capable of leading to the discovery of a law, and so they are analogous to many other facts which do not seem to us isolated, but closely grouped with others. The isolated fact attracts all eyes, those of the layman as well as of the scientist. But what the genuine physicist alone knows how to see, is the bond which unites many facts whose analogy is profound but hidden. The story of Newton's apple is probably not true, but it is symbolic; let us speak of it then as if it were true. Well then, we must believe that before Newton plenty of men had seen apples fall; not one knew how to conclude anything therefrom. Facts would be sterile were there not minds capable of choosing among them, discerning those behind which something was hidden, and of recognizing what is hiding, minds which under the crude fact perceive the soul of the fact.

We find just the same thing in mathematics. From the varied elements at our disposal we can get millions of different combinations; but one of these combinations, in so far as it is isolated, is absolutely void of value. Often we have taken great pains to construct it, but it serves no purpose, if not perhaps to furnish a task in secondary education. Quite otherwise will it be when this combination shall find place in a class of analogous combinations and we shall have noticed this analogy. We are no longer in the presence of a fact but of a law. And upon that day the real discoverer will not be the workman who shall have patiently built up certain of these combinations; it will be he who brings to light their kinship. The first will

[2] See "The Choice of Facts," *The Monist*, April, 1909.

have seen merely the crude fact, only the other will have perceived the soul of the fact. Often to fix this kinship it suffices him to make a new word, and this word is creative. The history of science furnishes us a crowd of examples familiar to all.

The celebrated Vienna philosopher Mach has said that the rôle of science is to produce economy of thought, just as machines produce economy of effort. And that is very true. The savage reckons on his fingers or by heaping pebbles. In teaching children the multiplication table we spare them later innumerable pebble bunchings. Some one has already found out with pebbles or otherwise, that 6 times 7 is 42 and has had the idea of noting the result, and so we need not do it over again. He did not waste his time even if he reckoned for pleasure: his operation took him only two minutes; it would have taken in all two milliards if a milliard men had had to do it over after him.

The importance of a fact then is measured by its yield, that is to say, by the amount of thought it permits us to spare.

In physics the facts of great yield are those entering into a very general law, since from it they enable us to foresee a great number of others, and just so it is in mathematics. Suppose I have undertaken a complicated calculation and laboriously reached a result: I shall not be compensated for my trouble if thereby I have not become capable of foreseeing the results of other analogous calculations and guiding them with a certainty that avoids the gropings to which one must be resigned in a first attempt. On the other hand I shall not have wasted my time if these gropings themselves have ended by revealing to me the profound analogy of the problem just treated with a much more extended class of other problems; if they have shown me at once the resemblances and differences of these, if in a word they have made me perceive the possibility of a

generalization. Then it is not a new result I have won, it is a new power.

The simple example that comes first to mind is that of an algebraic formula which gives us the solution of a type of numeric problems when finally we replace the letters by numbers. Thanks to it a single algebraic calculation saves us the pains of ceaselessly beginning over again new numeric calculations. But this is only a crude example; we all know there are analogies inexpressible by a formula and all the more precious.

A new result is of value, if at all, when in unifying elements long known but hitherto separate and seeming strangers one to another, it suddenly introduces order where apparently disorder reigned. It then permits us to see at a glance each of these elements and its place in the assemblage. This new fact is not merely precious by itself, but it alone gives value to all the old facts it combines. Our mind is weak as are the senses; it would lose itself in the world's complexity were this complexity not harmonious; like a near-sighted person it would see only the details and would be forced to forget each of these details before examining the following, since it would be incapable of embracing all. The only facts worthy our attention are those which introduce order into this complexity and so make it accessible.

Mathematicians attach great importance to the elegance of their methods and their results. This is not pure dilettantism. What is it indeed that gives us the feeling of elegance in a solution, in a demonstration? It is the harmony of the diverse parts, their symmetry, their happy balance; in a word it is all that introduces order, all that gives unity, that permits us to see clearly and to comprehend at once both the *ensemble* and the details. But this is exactly what yields great results; in fact the more we see this aggregate clearly and at a single giance, the better

we perceive its analogies with other neighboring objects, consequently the more chances we have of divining the possible generalizations. Elegance may produce the feeling of the unforeseen by the unexpected meeting of objects we are not accustomed to bring together; there again it is fruitful, since it thus unveils for us kinships before unrecognized. It is fruitful even when it results only from the contrast between the simplicity of the means and the complexity of the problem set; it makes us then think of the reason for this contrast and very often makes us see that chance is not the reason; that it is to be found in some unexpected law. In a word, the feeling of mathematical elegance is only the satisfaction due to any adaptation of the solution to the needs of our mind, and it is because of this very adaptation that this solution can be for us an instrument. Consequently this esthetic satisfaction is bound up with the economy of thought. Again the comparison of the Erechtheum comes to my mind, but I must not use it too often.

It is for the same reason that, when a rather long calculation has led to some simple and striking result, we are not satisfied until we have shown that we should have been *able to foresee,* if not this entire result, at least its most characteristic traits. Why? What prevents our being content with a calculation which has told us, it seems, all we wished to know? It is because, in analogous cases, the long calculation might not again avail, and that this is not so about the reasoning often half intuitive which would have enabled us to foresee. This reasoning being short, we see at a single glance all its parts, so that we immediately perceive what must be changed to adapt it to all the problems of the same nature which can occur. And then it enables us to foresee if the solution of these problems will be simple, it shows us at least if the calculation is worth undertaking.

UMI

What we have just said suffices to show how vain it would be to seek to replace by any mechanical procedure the free initiative of the mathematician. To obtain a result of real value, it is not enough to grind out calculations, or to have a machine to put things in order; it is not order alone, it is unexpected order which is worth while. The machine may gnaw on the crude fact, the soul of the fact will always escape it.

Since the middle of the last century, mathematicians are more and more desirous of attaining absolute rigor; they are right, and this tendency will be more and more accentuated. In mathematics rigor is not everything, but without it there is nothing. A demonstration which is not rigorous is nothingness. I think no one will contest this truth. But if it were taken too literally, we should be led to conclude that before 1820, for example, there was no mathematics; this would be manifestly excessive; the geometers of that time understood voluntarily what we explain by prolix discourse. This does not mean that they did not see it at all; but they passed over too rapidly, and to see it well would have necessitated taking the pains to say it.

But is it always needful to say it so many times; those who were the first to emphasize exactness before all else have given us arguments that we may try to imitate; but if the demonstrations of the future are to be built on this model, mathematical treatises will be very long; and if I fear the lengthenings, it is not solely because I deprecate encumbering libraries, but because I fear that in being lengthened out, our demonstrations may lose that appearance of harmony whose usefulness I have just explained.

The economy of thought is what we should aim at, so it is not enough to supply models for imitation. It is needful for those after us to be able to dispense with these models and in place of repeating an argument already

made, summarize it in a few words. And this has already been attained at times. For instance there was a type of reasoning found everywhere, and everywhere alike. They were perfectly exact, but long. Then all at once the phrase "uniformity of convergence" was hit upon, and this phrase made those arguments needless; we were no longer called upon to repeat them, since they could be understood. Those who conquer difficulties then do us a double service; first they teach us to do as they at need, but above all they enable us as often as possible to avoid doing as they, yet without sacrifice of exactness.

We have just seen by one example the importance of words in mathematics, but many others could be cited. It is hard to believe how much a well-chosen word can economize thought, as Mach says. Perhaps I have already said somewhere that mathematics is the art of giving the same name to different things. It is fitting that these things, differing in matter, may be alike in form, that they may, so to speak, run in the same mould. When the language has been well chosen, we are astonished to see that all the proofs made for a certain object apply immediately to many new objects; there is nothing to change, not even the words, since the names have become the same.

A well-chosen word usually suffices to do away with the exceptions from which the rules stated in the old way suffer; this is why we have created negative quantities, imaginaries, points at infinity, and what not. And exceptions, we must not forget, are pernicious because they hide the laws.

Well, this is one of the characteristics by which we recognize the facts which yield great results. They are those which allow of these happy innovations of language. The crude fact then is often of no great interest; we may point it out many times without having rendered great service to science. It takes value only when a wiser thinker per-

UMI

ceives the relation for which it stands, and symbolizes it by a word.

Moreover the physicists do just the same. They have invented the word "energy," and this word has been prodigiously fruitful, because it also made the law by eliminating the exceptions, since it gave the same name to things differing in matter and like in form.

Among words that have had the most fortunate influence I would select "group" and "invariant." They have made us see the essence of many mathematical reasonings; they have shown us in how many cases the old mathematicians considered groups without knowing it, and how, believing themselves far from one another, they suddenly found themselves near without knowing why.

To-day we should say that they had dealt with isomorphic groups. We now know that in a group the matter is of little interest, the form alone counts, and that when we know a group we thus know all the isomorphic groups; and thanks to these words "group" and "isomorphism," which condense in a few syllables this subtile rule and quickly make it familiar to all minds, the transition is immediate and can be done with every economy of thought effort. The idea of group besides attaches to that of transformation. Why do we put such a value on the invention of a new transformation? Because from a single theorem it enables us to get ten or twenty; it has the same value as a zero adjoined to the right of a whole number.

This then it is which has hitherto determined the direction of mathematical advance, and just as certainly will determine it in the future. But to this end the nature of the problems which come up contributes equally. We can not forget what must be our aim. In my opinion this aim is double. Our science borders upon both philosophy and physics, and we work for our two neighbors; so we have

always seen and shall still see mathematicians advancing in two opposite directions.

On the one hand, mathematical science must reflect upon itself, and that is useful since reflecting on itself is reflecting on the human mind which has created it, all the more because it is the very one of its creations for which it has borrowed least from without. This is why certain mathematical speculations are useful, such as those devoted to the study of the postulates, of unusual geometries, of peculiar functions. The more these speculations diverge from ordinary conceptions, and consequently from nature and applications, the better they show us what the human mind can create when it frees itself more and more from the tyranny of the external world, so the better they let us know it in itself.

But it is toward the other side, the side of nature that we must direct the bulk of our army. There we meet the physicist or the engineer, who says to us: "Please integrate this differential equation for me; I might need it in a week in view of a construction which should be finished by that time." "This equation," we answer, "does not come under one of the integrable types; you know there are not many." "Yes, I know; but then what good are you?" Usually to understand each other is enough; the engineer in reality does not need the integral in finite terms; he needs to know the general look of the integral function, or he simply wants a certain number which could readily be deduced from this integral if it were known. Usually it is not known, but the number can be calculated without it if we know exactly what number the engineer needs and with what approximation.

Formerly an equation was considered solved only when its solution had been expressed by aid of a finite number of known functions; but that is possible scarcely once in a hundred times. What we always can do, or rather what

UMI

we should always seek to do, is to solve the problem *qualitatively* so to speak; that is to say, seek to know the general form of the curve which represents the unknown function.

It remains to find the *quantitative* solution of the problem; but if the unknown cannot be determined by a finite calculation, it may always be represented by a convergent infinite series which enables us to calculate it. Can that be regarded as a true solution? We are told that Newton sent Leibnitz an anagram almost like this: *aaaaabbbeeeeii*, etc. Leibnitz naturally understood nothing at all of it; but we, who have the key, know that this anagram meant translated into modern terms: "I can integrate all differential equations"; and we are tempted to say that Newton had either great luck or strange delusions. He merely wished to say he could form (by the method of indeterminate coefficients) a series of powers formally satisfying the proposed equation.

Such a solution would not satisfy us to-day, and for two reasons: because the convergence is too slow and because the terms follow each other without obeying any law. On the contrary, the series Θ seems to us to leave nothing to be desired, first because it converges very quickly (this is for the practical man who wishes to get at a number as quickly as possible) and next because we see at a glance the law of the terms (this is to satisfy the esthetic need of the theorist).

But then there are no longer solved problems and others which are not; there are only problems *more or less* solved according as they are solved by a series converging more or less rapidly, or ruled by a law more or less harmonious. It often happens however that an imperfect solution guides us toward a better one. Sometimes the series converges so slowly that the computation is impracticable and we have only succeeded in proving the possibility of the problem.

And then the engineer finds this a mockery, and justly, since it will not aid him to complete his construction by the date fixed. He little cares to know if it will benefit engineers of the twenty-second century. But as for us, we think differently and we are sometimes happier to have spared our grand-children a day's work than to have saved our contemporaries an hour.

Sometimes by groping, empirically so to speak, we reach a formula sufficiently convergent. "What more do do you want?" says the engineer. And yet, in spite of all, we are not satisfied; we should have liked *to foresee* that convergence. Why? Because if we had known how to foresee it once, we would know how to foresee it another time. We have succeeded; that is a small matter in our eyes if we cannot validly expect to do so again.

In proportion as science develops its total, comprehension becomes more difficult; then we seek to cut it in pieces and to be satisfied with one of these pieces: in a word, to specialize. If we went on in this way, it would be a grievous obstacle to the progress of science. As we have said, it is by unexpected unions between its diverse parts that it progresses. To specialize too much would be to forbid these drawings together. It is to be hoped that congresses like those of Heidelberg and Rome, by putting us in touch with one another will open for us vistas over neighboring domains and oblige us to compare them with our own, to range somewhat abroad from our own little village; thus they will be the best remedy for the danger just mentioned.

But I have lingered too long over generalities, it is time to enter into detail.

Let us pass in review the various special sciences which combined make mathematics; let us see what each has accomplished, whither it tends and what we may hope from it. If the preceding views are correct, we should see

that the greatest advances in the past have happened when two of these sciences have united, when we have become conscious of the similarity of their form, despite the difference of their matter, when they have so modeled themselves upon each other that each could profit by the other's conquests. We should at the same time foresee in combinations of the same sort, the progress of the future.

ARITHMETIC.

Progress in arithmetic has been much slower than in algebra and analysis, and it is easy to see why. The feeling of continuity is a precious guide which the arithmetician lacks; each whole number is separated from the others, —it has, so to speak, its own individuality. Each of them is a sort of exception and this is why general theorems are rarer in the theory of numbers; this is also why those which exist are more hidden and longer elude the searchers.

If arithmetic is behind algebra and analysis, the best thing for it to do is to seek to model itself upon these sciences so as to profit by their advance. The arithmetician ought therefore to take as guide the analogies with algebra. These analogies are numerous and if, in many cases, they have not yet been studied sufficiently closely to become utilizable, they at least have long been foreseen, and even the language of the two sciences shows they have been recognized. Thus we speak of transcendent numbers and thus we account for the future classification of these numbers already having as model the classification of transcendent functions, and still we do not as yet very well see how to pass from one classification to the other; but had it been seen, it would already have been accomplished and would no longer be the work of the future.

The first example that comes to my mind is the theory of congruences where is found a perfect parallelism to the

theory of algebraic equations. Surely we shall succeed in completing this parallelism, which must hold for instance between the theory of algebraic curves and that of congruences with two variables. And when the problems relative to congruences with several variables shall be solved, this will be a first step toward the solution of many questions of indeterminate analysis.

ALGEBRA.

The theory of algebraic equations will still long hold the attention of geometers; numerous and very different are the sides whence it may be attacked.

We need not think algebra is ended because it gives us rules to form all possible combinations; it remains to find the interesting combinations, those which satisfy such and such a condition. Thus will be formed a sort of indeterminate analysis where the unknowns will no longer be whole numbers, but polynomials. This time it is algebra which will model itself upon arithmetic, following the analogy of the whole number to the integral polynomial with any coefficients or to the integral polynomial with integral coefficients.

GEOMETRY.

It looks as if geometry could contain nothing which is not already included in algebra or analysis; that geometric facts are only algebraic or analytic facts expressed in another language. It might then be thought that after our review there would remain nothing more for us to say relating specially to geometry. This would be to fail to recognize the importance of well constructed language, not to comprehend what is added to the things themselves by the method of expressing these things and consequently of grouping them.

First the geometric considerations lead us to set our-

UMI

selves new problems; these may be, if you choose, analytic problems, but such as we never would have set ourselves in connection with analysis. Analysis profits by them however, as it profits by those it has to solve to satisfy the needs of physics.

A great advantage of geometry lies in the fact that in it the senses can come to the aid of thought, and help find the path to follow, and many minds prefer to put the problems of analysis into geometric form. Unhappily our senses cannot carry us very far, and they desert us when we wish to soar beyond the classical three dimensions. Does this mean that, beyond the restricted domain wherein they seem to wish to imprison us, we should rely only on pure analysis and that all geometry of more than three dimensions is vain and objectless? The greatest masters of a preceding generation would have answered "yes"; to-day we are so familiarized with this notion that we can speak of it, even in a university course, without arousing too much astonishment.

But what good is it? That is easy to see: First it gives us a very convenient terminology, which expresses concisely what the ordinary analytic language would say in prolix phrases. Moreover, this language makes us call like things by the same name and emphasize analogies it will never again let us forget. It enables us therefore still to find our way in this space which is too big for us and which we cannot see, always recalling visible space which is only an imperfect image of it doubtless, but which is nevertheless an image. Here again, as in all the preceding examples, it is analogy with the simple which enables us to comprehend the complex.

This geometry of more than three dimensions is not a simple analytic geometry; it is not purely quantitative, but qualitative also, and it is in this respect above all that it becomes interesting. There is a science called *Analysis*

Situs and which has for its object the study of the positional relations of the different elements of a figure, apart from their sizes. This geometry is purely qualitative; its theorems would remain true if the figures, instead of being exact, were roughly imitated by a child. We may also make an *Analysis Situs* of more than three dimensions. The importance of *Analysis Situs* is enormous and cannot be too much emphasized; the advantage obtained from it by Riemann, one of its chief creators, would suffice to prove this. We must achieve its complete construction in the higher spaces; then we shall have an instrument which will enable us really to see in hyperspace and supplement our senses.

The problems of *Analysis Situs* would perhaps not have suggested themselves if the analytic language alone had been spoken; or rather, I am mistaken, they would have occurred surely, since their solution is essential to a crowd of questions in analysis, but they would have come singly, one after another, and without our being able to perceive their common bond.

CANTORISM.

I have spoken above of our need to go back continually to the first principles of our science, and of the advantage of this for the study of the human mind. This need has inspired two endeavors which have taken a very prominent place in the most recent annals of mathematics. The first is Cantorism which has rendered our science such conspicuous service. Cantor introduced into science a new way of considering mathematical infinity. One of the characteristic traits of Cantorism is that in place of going up to the general by building up constructions more and more complicated and defining by construction, it starts from the *genus supremum* and defines only, as the scholastics would have said, *per genus proximum et differentiam spe-*

cificam. Thence comes the horror it has sometimes inspired in certain minds, for instance in Hermite, whose favorite idea was to compare the mathematical to the natural sciences. With most of us these prejudices have been dissipated, but it has come to pass that we have encountered certain paradoxes, certain apparent contradictions that would have delighted Zeno the Eleatic and the school of Megara. And then each must seek the remedy. For my part, I think, and I am not the only one, that the important thing is never to introduce entities not completely definable in a finite number of words. Whatever be the cure adopted, we may promise ourselves the joy of the doctor called in to follow a beautiful pathologic case.

THE INVESTIGATION OF THE POSTULATES.

On the other hand, efforts have been made to enumerate the axioms and postulates, more or less hidden, which serve as foundation to the different theories of mathematics. Professor Hilbert has obtained the most brilliant results. It seems at first that this domain would be very restricted and there would be nothing more to do when the inventory should be ended, which could not take long. But when we shall have enumerated all, there will be many ways of classifying all; a good librarian always finds something to do, and each new classification will be instructive for the philosopher.

Here I end this review which I could not dream of making complete. I think these examples will suffice to show by what mechanism the mathematical sciences have made their progress in the past and in what direction they must advance in the future.

H. POINCARÉ.

PARIS, FRANCE.

TRANSFINITE NUMBERS AND THE PRINCIPLES OF MATHEMATICS.

PART I.

One result of Georg Cantor's discovery of the transfinite cardinal and ordinal numbers has been the development of more satisfactory views on the principles of mathematics. To this end, also, the symbolic logic of Peano, Frege, and Russell[1] contributed by enabling one, for the first time, to reach precision in such subjects as the relation of logic to mathematics, and the meaning of "definition" and "existence."

In this first part, I give an account of these things, and, in the second part, I will review the modifications in logic and in our views of the principles of mathematics which progress in the theory of aggregates has necessitated. I hope to show that, just as we have been forced, especially during the nineteenth century, to a more rigorous foundation of the methods and results of mathematical analysis, so we are forced to logical investigations by that development of mathematics to which I have just referred.

In this article I wish to emphasize an aspect in the development of views on the principles of mathematics other than that of the gradual *rapprochement* of mathematics and logic and their final reconciliation owing to the good offices of the logic of relations as promulgated by De Morgan, C. S. Peirce, Schröder, Dedekind, Frege, Peano, and Russell. I wish to point out the service which the theory of transfinite numbers has done first, in drawing atten-

[1] This symbolic logic is a great advance on the older symbolic logic, of which Schröder has given an excellent account (*Vorlesungen über die Algebra der Logik*, 3 volumes, Leipsic, 1890 and subsequent years; part of the third volume is not yet published).

UMI

tion to what are known as "the contradictions of the theory of aggregates," and hence to the necessity for a remoulding of logic; and, secondly, in clearly separating cardinal and ordinal numbers in our minds, and so making us, by analogy, more precise in our distinctions of signless integers and positive integers, of the integer n and the ratio $n:1$, and so on. These distinctions have made us give up the "principle of permanence," which formerly played such a great part in mathematics, for we are compelled to admit that it consists in identifying things whose difference is clearly discernible.

Thus, the advance of mathematics has brought it nearer and nearer to logic; the extent of the validity of mathematical conceptions and methods has been examined ever more closely; and it is not difficult to see that, by this, we have attained to a more thorough knowledge, and even, by the capacity which we have gained of avoiding those pseudo-problems to which methods extended beyond their domain of validity give rise, to a *practical* advance.

I.

Cantor was led to see the necessity for introducing certain definitely infinite numbers by his mathematical researches on infinite aggregates of points situated on a finite line (using a geometrical terminology for conceptions which are, in reality, purely arithmetical); but, logically, the theory is independent of this origin, and here[2] I will give the independent grounds on which, in the *Grundlagen,* Cantor made the introduction of these numbers rest.

Among the finite integers $1, 2, \ldots, \nu, \ldots$ there is no greatest, but, although it would be contradictory to speak of a greatest finite integer μ (for there is always a greater one $\mu+1$), there is no contradiction[3] involved in introducing a *new,* non-finite number (ω), which is defined as the *first* number that follows *all* the numbers $1, 2, \ldots, \nu, \ldots$ (in their order of magnitude). The

[2] A very full historical account by me appeared in the *Archiv der Math. und Phys.* for 1906 and 1909, and the rest will appear shortly.

[3] This is the point which will be found to require for its adequate discussion, all the resources of logic (see below).

interest that attaches to the introduction of a series of such "transfinite" numbers, the first ones of which Cantor has denoted:

$$\omega, \omega+1, \omega+2, \ldots, \omega+\nu, \ldots \omega\cdot 2, \omega\cdot 2+1, \ldots \omega\cdot\nu, \ldots\omega^2,\ldots$$
$$\omega^\nu, \ldots\omega^\omega, \ldots\omega^{\omega^\omega}, \ldots, \ldots, (N)$$

ν being any finite integer, is, of course, to be seen from the history of those mathematical questions which necessitated the introduction of these numbers;[4] but here we are only concerned with the question whether the conception of such numbers is logically possible, that is to say, leads to no contradiction.[5] That Cantor, to most intents and purposes, showed this by his above introduction and subsequent definition of ω, is true, and, further, he successfully classified and answered the objections made by philosophers and mathematicians, from the time of Aristotle, against the actual (or *completed*, as distinguished from the "potential" or "becoming") infinite.[6] A characteristic and illuminating example of this criticism was given *à propos* of Dühring's arguments against the actual infinite (*Eigentlich-Unendlich*).[7] These arguments can, said Cantor, be reduced, either to the statement that a definite finite number, however large, can never be infinite (a statement which is a truism) or that a variable unlimitedly great finite number can not be thought of with the predicate of definiteness, and hence also not with the predicate of being (which again immediately results from the essence of variability). To conclude, as Dühring does, the non-thinkability of definitely infinite numbers is like arguing that, because there are innumerable intensities of green, there can be no red.[8]

[4] The use of transfinite numbers in important questions of mathematics has been shown, for example, by G. H. Hardy (*Proc. Lond. Math. Soc.* (2), vol. I, 1904, pp. 285-290) and myself (*Mess. of Math.*, April, 1904, pp. 166-171, and Crelle's *Journ. für Math.*, Bd. CXXVIII, 1905, pp. 169-210).

[5] Cantor (*Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, Leipsic, 1883, pp. 18-20), maintained the thesis that the formation of concepts in mathematics is completely *free*, and has only to satisfy the condition of the logical consistency of these concepts with one another. Such concepts then have "existence" (in mathematics). Cf. below on the question whether "freedom from contradiction" is necessary or sufficient for the "existence" of a concept.

[6] *Grundlagen*, pp. 9-18, 43-46; *Zur Lehre vom Transfiniten*, Halle a. S., 1890 (reprint of Cantor's articles in the *Zeitschr. f. Phil. u. philos. Kritik*, Bde. LXXXVIII, XCI, and XCII, 1885-1887).

[7] See *Grundlagen*, pp. 44-45.

[8] The arguments against the infinite in mathematics have also been discussed exhaustively by Couturat (*De l'infini mathématique*, Paris, 1896, pp.

The logically exact investigation as to the existence of numbers defined by an infinite process (as ω is by the finite numbers, or an irrational number by the rationals) was begun by Russell, and I return to the question in the next section.

The series of the transfinite numbers was, now, shown by Cantor to fall into certain divisions, which he called "number-classes"; which are characterized by the property that, if α and β are any numbers of the same class, all the numbers (from 1 on) preceding α can be brought (in a different order, of course) into a correspondence,[9] which is one-one, with all those preceding β; and inversely Cantor expressed this by saying that the first class of numbers had the same "power" as the second, or that one, and only one, "power" belonged to each "number-class."

Thus, in addition to the series of finite and transfinite (ordinal) numbers, there is a series of finite and transfinite powers; for finite aggregates the conceptions of power and (ordinal) number appear to coincide,[10] and such an aggregate has always the same number, however it may be arranged; but a given infinite aggregate, though no re-arrangement can alter its power, since this attribute is, by the definition, independent of order, can have various (ordinal) numbers,—in fact, any number of a certain class,—according to the way in which it is arranged.

But, even when an aggregate is "simply ordered" (that is to say, when an "order" is given to the terms of an aggregate such that, if *a* and *b* be any two terms, *a* either *precedes* or *follows b* in virtue of some relation, not necessarily in order of space or time), it need not have an ordinal number. In fact, Cantor's ordinal numbers only apply to certain kinds of ordered aggregates, which he called "well-ordered," and which are characterized by the property that any selection of terms has, in the order of the original series, an element of lowest rank. Thus, the series

441-503) and by Russell (*The Principles of Mathematics,* vol. I, Cambridge, 1903, pp. 355-362).

[9] See below.

[10] However, strictly speaking they do not *coincide.* The point is the same as the one about signless integers (classes) and positive integers (relations) referred to below.

$$a_1, a_2, \ldots, a_\nu, \ldots; b_1, b_2$$

is well-ordered, but not the series

$$b_1, b_2; \ldots, a_\nu, \ldots, a_2, a_1,$$

where ν is any finite number and the dots indicate that *all* the a_ν's, where ν is finite, occur in the order shown. Accordingly, Cantor generalized and renamed his fundamental concepts in the theory of transfinite numbers as follows:[11]

"By an 'aggregate' or 'manifold' (*Menge*), we understand any collection by the mind (*Zusammenfassung*) M of definite well-distinguished objects m of our intuition or of our thought (which are called the 'elements' of M) to a whole.

"Every aggregate M has a definite 'power,' which we also call 'cardinal number.'

"We call 'power' (*Mächtigkeit*) or 'cardinal number' of M the general concept which, by means of our active faculty of thought, is obtained from the aggregate M by abstracting from the nature (*Beschaffenheit*) of its different elements m and from the order in which they are given."

Cantor proved that, in order that two aggregates, M and N, should have the same cardinal number, it is necessary and sufficient that they should be "equivalent"[12] (*äquivalent*), that is to say, that there should be a one-one correspondence between the elements m and the elements n. The operations of addition, multiplication, and exponentiation for cardinal numbers were then defined,[13] and certain other questions of mathematical importance investigated, including a short treatment of the *finite* cardinal numbers[14] and the smallest transfinite cardinal number ($\aleph_0$).[15] But also, what concerns us intimately at present, Cantor also mentioned a series of

[11] *Math. Ann.*, Bd. XLVI (1895), pp. 481-512; Bd. XLIX (1897), pp. 207-246.

[12] Russell has used the word "similar" instead of "equivalent" and "like" instead of "similar" (Cantor's *ähnlich*, see below); while Dedekind used *ähnlich* where Cantor used *äquivalent*. At Dr. Carus's suggestion, we follow Cantor's terminology here.

[13] *Math. Ann.*, XLVI, pp. 485-488.

[14] *Ibid.*, pp. 489-492.

[15] *Ibid.*, pp. 492-495.

UMI

cardinal numbers ascending in magnitude and such that there is no cardinal number between two consecutive terms of the series:

$$\aleph_0, \aleph_1, \aleph_2, \cdots, \aleph_\nu, \cdots \aleph_\omega, \aleph_{\omega+1}, \cdots, \aleph_\gamma, \cdots; \text{ (A)}$$

as a subject for future investigation,[16] and implied that every transfinite cardinal number is to be found in this series. The conception of an "ordered aggregate" was then introduced:[17]

"We call an aggregate M 'simply ordered,' if a definite 'order of precedence' (*Rangordnung*) rules its elements m, so that of any two elements m_1 and m_2 one takes the 'lower' and the other the 'higher' rank; and so that if of three elements m_1, m_2, and m_3, m_1 is lower than m_2 and m_2 lower than m_3, m_1 is always lower than m_3." Such orders are order of magnitude and order of succession in time. Evidently, we are presupposed to have the idea of such a relation in general and it is not defined.

"Every (simply-) ordered aggregate has a definite *'ordinal type,'* by which we understand the general concept which results from M, when we abstract from the nature of the elements m, but retain the order of precedence among them." That two ordered aggregates should have the same type, it is necessary and sufficient they should be "similar" (*ähnlich*); that is to say, that there should be a one-one correspondence such that the order of precedence of corresponding elements is kept.

An important case of a simply-ordered aggregate is a "well-ordered aggregate,[18] which has been characterized above. The types of well-ordered aggregates were, now, called "ordinal numbers," and thus we arrive at the series (N). Now, the cardinal numbers of the various "segments"[19] of this series (N) form the series (A), which is such that there is no cardinal number which lies, in magnitude, between two consecutive Alephs, and none less than any one (for example consider $\aleph_\omega$) which is not one of the Alephs preceding

[16] *Ibid.*, pp. 495, 484.

[17] *Ibid.*, pp. 496-498.

[18] See the article in *Math. Ann.*, Bd. XLIX.

[19] The "segment" defined by the term a of a well-ordered series is the series of all terms preceding a. Cantor used the word *Abschnitt* (*Math. Ann.*, Bd. XLIX, p. 210).

that one in (A). Further, (A) possesses the remarkable property of being similar to (N). The other investigations of Cantor on ordinal numbers are of more exclusively mathematical interest.

In the question as to the existence of the various cardinal numbers and ordinal types defined by Cantor, there was still an opportunity left for skepticism, and one of the chief objects of Russell's work[20] was so to define the numbers as to leave no doubt about their existence. We must, then, next give an account of that part of modern work on symbolic logic which is necessary for the comprehension of this object.

II.

Peano's logical calculus differs from the previous systems of algebra of logic[21] in one or both of the respects of being more convenient in symbolism and of containing more subtle distinctions between certain fundamental ideas. Thus, in the latter respect, Peano had the distinction, which was not possessed by Schröder[22]

[20] Cf. *op. cit.*, pp. ix, 111-116, 277-286, 313, 321-322, 497-498.

[21] Although Leibniz had worked out projects of an algebra of logic and a general symbolism, his work in this direction only began to be known when his manuscripts began to be published by. J. E. Erdmann in 1840. The work in this direction of Leibniz's successors—of whom the greatest was J. H. Lambert—made little impression, and it was George Boole and Augustus De Morgan, about the middle of the nineteenth century, who must be regarded as the true founders of what we now know as symbolic logic. A valuable work of an orthodox Boolian character, containing much careful historical research, is J. Venn's *Symbolic Logic*, London, 1880 (2d. ed., 1894); and the most complete works on the logic of Leibniz are: B. Russell, *A Critical Exposition of the Philosophy of Leibniz*, Cambridge, 1900; and L. Couturat, *La logique de Leibniz d'après des documents inédits*, Paris, 1901, and *Opuscules et fragments inédits de Leibniz*, Paris, 1903.

It must be mentioned that the introduction of "propositions containing variables" and of implication between them was first explicitly made by H. MacColl in 1878. Still MacColl did not observe, like Frege and Peano, that these notions made it possible to formulate all mathematical deductions in symbols—what was impossible with the traditional or Aristotelian logic—and indeed, as Russell has shown, rather confused the essential difference between these *propositional functions* and *propositions* proper. The logic of relations of De Morgan, Peirce, Schröder, Frege, Dedekind, Peano and Russell will be referred to afterwards.

[22] Cf. Schröder, *op. cit.*, Bd. II, 2. Abteilung, Leipsic, 1905, pp. 461, 597; *Verh. d. Math. Congr. in Zürich*, Leipsic, 1898, p. 154; G. Frege, *Kritische Beleuchtung einiger Punkte in E. Schröders Vorlesungen über die Algebra der Logik*, *Archiv für systemat. Phil.*, I, 1895, pp. 433-456; *Grundgesetze der Arith.*, I, Jena, 1893, p. 2; Russell, *Principles*, pp. 19, 78; Couturat, *Les Principes des mathématiques*, Paris, 1905, pp. 22-21 (a German translation of this book by C. Siegel was published at Leipsic in 1908 under the title: *Die philosophischen Prinzipien der Mathematik*).

Frege's work, which began in 1879, is of a far more subtle character than

or any other previous writer, between "the class (or individual) *a is a member* of the class *b*" and "the class *a is contained in* the class *b*"; the former was symbolized by Peano $a \epsilon b$, the latter by a different sign;[23] and the latter may be defined: $x \epsilon a$ implies, for every such x, $x \epsilon b$.

Again, Peano distinguished between a term (say x) of a class and the class (ιx) composed of that single term, treated the conception of the "variable" at some length, and so on.

While referring for more detailed accounts of Peano's system of writing all mathematical propositions in logical symbolism, which implies a calculus of logic, to other works,[24] we shall here notice more particularly some points in it and in Russell's work of great importance to us in our present subject.

* * *

When the propositions a, b, said Peano,[25] contain undetermined entities $x, y, \ldots$,[26] as they do in general, then the suffix $x, y, \ldots$ attached to the sign of implication between a and b makes the whole read: "a implies b, whatever $x, y, \ldots$ may be" (provided, of course, they satisfy the conditions that may have been imposed on them at the beginning), and if a and b contain two groups of undeter-

Peano's (cf. Russell, *op. cit.*, pp. 500-522), and consequently far more suited to the investigation of the principles of mathematics—for which purpose, indeed, his ideography was invented. His symbolism, however, is so cumbrous, that Russell, who, independently of Frege, arrived at many of Frege's points of view, combined Frege's ideas with Peano's symbolism (slightly modified) in his most recent work (*Amer. Journ. of Math.*, XXVIII, 1906; and XXX, 1908).

[23] Since Peano wrote $b \supset a$ for "b contains a," for "a is contained in b" he used a sign which is a deformation of an inverted C.

[24] See pp. 370-378 (on the symbolism of Peano and Russell) in Whitehead's Memoir *On Cardinal Numbers* (*Amer. Journ. of Math.*, Vol. XXIV, 1902, pp. 367-394); the references to the calculus of logic in the works of Russell and Couturat, and to Peano's logic in Russell, *op. cit.*, pp. 26-32, and Couturat, *op. cit.*, pp. 5, 6, 18, 24, 27; Peano's various *Formulaires* and the volumes of his *Rivista di matematica*; and Couturat's account of the work of Peano and his school in the *Bull. des sci. math.*, 2e série, t. XXV, 1901.

[25] *Arithmetices Principia nova methodo exposita*, Turin, 1889, p. viii; *Notations de logique mathématique*, Turin, 1894, pp. 16-18, 20-22.
On the subject of the variable, propositional functions, formal implication, individual and class, see Couturat, *op. cit.*, pp. 17, 21-23.

[26] We may restrict $x, y, \ldots$ to be real or imaginary numbers, points, classes, propositions,.... For example "x and y are (real or complex) numbers" implies (whatever numbers x and y may be) $(x+y)^2 = x^2 + 2xy + y^2$.

mined entities $x, y, \ldots$ and $u, v, \ldots$, and we wish to say that $u, v, \ldots$ are such that, whatever $x, y, \ldots$ may be, a implies b, then we write as suffixes only the entities $(x, y, \ldots)$ with respect to which we make the deduction. The resulting proposition is then a condition between $u, v, \ldots$, and is independent of $x, y, \ldots$.[27]

If the value of a formula does not depend on the undetermined entity in it, just as the value of a definite integral does not depend on the variable (x) of integration, it is not necessary to explain the signification of x, as was done above.

If p is a proposition containing a variable x, we denote the class of x's which satisfy p_x by $x \ni p_x$ and read it: "the x's *such that* p_x is true." If p_x contains other variables $u, v, \ldots$ besides x, $x \ni p_x$ denotes a class which is a function of $u, v, \ldots$, but independent of x.

III.

All the propositions of pure mathematics are, according to Russell,[28] of the form "p implies q," where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. Logical constants are all notions definable[29] in terms of the following: Implication, the relation of a term to a class of which it is a member (ϵ), the notion of *such that*, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics uses a notion which is not a constituent of the propositions which it considers, namely, the notion of truth. A proposition[30] is de-

[27] Thus, if x is a real number $(x \epsilon q)$, and we write $x \epsilon q$ implies, for any such v, $ax^2+bx+c=0$, the proposition is "a, b, c are such that, whatever x is, $ax^2+bx+c=0$." The implication *without* any index (which is equivalent to that with *all* the indices x, a, b, c) states the false proposition: "whatever numbers x, a, b, c are, $ax^2+bx+c=0$."

[28] *Op. cit.*, p. 3; cf. Couturat's book quoted—which may be described as a more popular exposition of Russell's work—pp. 1-6.

[29] For the meaning of this term, see below.

[30] The calculus of propositions (Russell, *op. cit.*, pp. 13-18, Couturat, *op. cit.*, pp. 8-16) must precede those of classes (Russell, *op. cit.*, pp. 18-23; Couturat, *op. cit.*, pp. 16-26) and of relations (Russell, *op. cit.*, pp. 23-26; Couturat,

finable as "that which implies itself,"[31] and must be distinguished clearly from what Peano (and Russell in the above statement) called "a proposition containing a variable," and Russell, in far preferable language, a "propositional function."[32] A proposition, we may say, is anything that is true or that is false. An expression such as "x is a man" is, therefore, not a proposition; but if we give to x any constant value whatever,[33] the expression becomes a proposition. This schematic form standing for any one of a whole class of propositions is called a "propositional function," and we may explain, but not define, this notion as follows: ϕx is a propositional function if, for every value of x, ϕx is a proposition, determinate when x is given. In this, x is called the *variable*, and we may say that a propositional function is, in general, true for some values of the variable and false for others.

When we say "x is a man implies x is mortal for all values of x," we have a genuine proposition, in which, though the letter x appears, it is absorbed in the same kind of way as the x under the integral sign in a definite integral, so that the result is no longer a function of x. In this case, x is what Peano called an "apparent variable," since the proposition does not depend upon the variable; whereas the variable was called "real" in propositional functions. Genuine propositions do not depend upon a variable or variables.[34]

op. cit., pp. 27-34), since the principles of the calculus of propositions are used in all reasoning.

On the calculus of classes, cf. the note on the theory of aggregates in Couturat, *op. cit.*, pp. 219-228.

The logic of relations, the mathematical importance of which was shown by Dedekind's application of it (Dedekind himself rediscovered much of it independently) and by Schröder's work, was, as Schröder rightly observed (*Verh. des ersten Math.-Congr. in Zürich, 1897*, Leipsic, 1898) somewhat neglected by Peano to the disadvantage of his logic. It was Russell (*Rev. de Math.*, VII) who first completed Peano's logic by a logic of relations in which the Peirce-Schröder ideas were modified so as to fit in with a logic which comprised more subtle distinctions than that of Schröder. Cf. Couturat, *op. cit.*, pp. 27-28.

Cf also Frege, *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Halle a. S., 1879 (Frege's work on ideography), pp. 15-24; *Funktion und Begriff*, Jena, 1891; *Grundgesetze*, I, 1893, pp. 5-25.

[31] Russell, *op. cit.*, p. 15.

[32] *Op. cit.*, pp. 12-13, 19-20; cf. Couturat, *op. cit.*, pp. 17-18.

[33] Such as "Socrates," "Plato," "the number 2."

[34] On the notion of the "variables" (the presence of which is marked by

In pure mathematics we assert what Russell[35] called the *formal implication*[36]: "$\phi(x, y, \ldots)$ implies $\psi(x, y, \ldots)$, whatever values $x, y, \ldots$ may have"; but we do not assert either ϕ or ψ of the entities $x, y, \ldots$; whilst, in applied mathematics, results which have been shown by pure mathematics to follow from some hypothesis are actually asserted of some constant satisfying the hypothesis in question. Thus terms which were variables become constant, and a new premise is always required, namely: this particular entity satisfies the hypothesis in question.[37]

The values of x "such that" ϕx is true form a class, and Russell[38] defined a class as all the terms satisfying some propositional function. That some limitation was required in this statement was recognized by Russell himself, in consequence of the contradiction discovered by him;[39] and this limitation, which forms indeed the kernel of our investigations, will be discussed at length hereafter.

IV.

The treatment of the meanings which can be attached to the word "definition" by Peano[40] and Burali-Forti prepared the way for a thoroughly satisfactory theory.[41]

The simplest form of a definition is, in Peano's symbolism,

$$x = a \qquad \mathrm{D}f.,$$

where x is a sign which has not, as yet, a signification, a is a group of signs having a known signification, and the sign of equality followed by "Df" — note that "$=\mathrm{D}f$" is one sign — indicates that we agree, for the sake of brevity, because the group

the occurrence of the words *any* or *some*, and may take any values) and "constants" in logic, see Russell, *op. cit.*, pp. 5-8, 89-94; Couturat, *op. cit.*, pp. 21-24; and Frege, "Was ist eine Funktion?" *Boltzmann-Festschrift*, 1904, pp. 656-666.

[35] *Op. cit.*, p. 5.

[36] As distinguished from the *material implication* (*op. cit.*, p. 14) between genuine propositions.

[37] Russell, *op. cit*, p. 8; cf. Couturat, *op. cit.*, pp. 4-5.

[38] *Op. cit.*, p. 20.

[39] *Op. cit.*, pp. 366-368, 101-107.

[40] *Notations*, pp. 44-51; *Les définitions mathématiques* (Bibl. du Congrès Internat. de Phil., III, "Logique et histoire des sciences," Paris, 1901, pp. 279-288).

[41] On definitions see also Frege, *Grundgesetze*, I, 43-52; II (1902), 69-80.

a denotes an important concept, to write the simple sign x instead of the group a.[42] Sometimes what we define is not a simple sign, but a group of signs, between which there are new signs, or a group of signs which have a signification separately, but such that their aggregate has not yet a signification. Then the definition follows an hypothesis (h) and has the form:

$$h \text{ implies that } x=a \qquad \text{D}f.^{43}$$

There are ideas, which we obtain by abstraction, which cannot be defined under the above form.[44] Let u be an object; by abstraction we deduce a new object ϕu; we cannot form an equality:

$$\phi u = \text{known expression} \qquad \text{D}f.,$$

for ϕu is an object of nature different from all those considered hitherto. Then we define *equality*[45] as follows:

$$h_{u,v} \text{ implies: } \phi u = \phi v .=. p_{u,v} \qquad \text{D}f.,^{46}$$

where $h_{u,v}$ is the hypothesis on the objects u and v, and $\phi u = \phi v$ is the equality which we define as meaning the same thing as $p_{u,v}$, a relation, with a known meaning, between u and v, which must satisfy the three conditions of being:

[42] Thus, Russell said (*op. cit.*, p. 429): "What distinguishes other branches of mathematics from logic is merely complication, which usually takes the form of an hypothesis that the variable belongs to some rather complicated class. Such a class will usually be denoted by a single symbol; and that the statement of the class in question is to be represented by such and such a symbol is what mathematicians call a *definition*. That is to say, a definition is no part of mathematics at all, and does not make any statement concerning the entities dealt with by mathematics, but is simply and solely a statement of a symbolic abbreviation: it is a proposition concerning symbols, not concerning what is symbolized." As regards the *philosophical* meaning of "definition," see *op. cit.*, pp. 15, 27, 111-112. Also (*op. cit.*, p. 15): "In the mathematical sense, a new propositional function is said to be defined when it is stated to be equivalent to (i. e., to imply and be implied by) a propositional function which has either been accepted as indefinable or has been defined in terms of indefinables." Cf. Couturat, *op. cit.*, pp. 10, 36-37.

[43] For example, in the definition of e^x as a power-series, the hypothesis is that x is a (real or complex) number.

[44] Peano, *Notations*, pp. 45-49.

[45] See the fifth section.

[46] Peano used dots (., :, ∴, ::) to separate the parts of a proposition, and the main implication of a proposition is always that immediately preceded or followed by the greatest collection (in one place) of dots. Further, dots between propositions are a sign of joint assertion or "logical multiplication" (p and q). Thus, in the proposition (if the letters denote propositions): p implies $q.x$: implies.r or s ∴ implies.t, the part p to s is the hypothesis (analyzable into an hypothesis "p implies q, and the proposition x is asserted" and the protasis "r or s",—a "logical addition") and t is the protasis (cf. *Notations*, pp. 11-13).

1. *Reflexive*; that is to say, $\phi u = \phi u$ or $p_{u,u}$ is true, whatever u is;

2. *Symmetrical*; that is to say, $\phi u = \phi v$ implies $\phi v = \phi u$, or $p_{u,v}$ implies $p_{v,u}$;

3. *Transitive*; that is to say, $\phi u = \phi v$ and $\phi v = \phi w$ imply that $\phi u = \phi w$, or $p_{u,v}$ and $p_{v,w}$ imply $p_{u,w}$.[47]

Among his examples, Peano gave[48] Stolz's[49] definition of a rational number. If a and b are natural numbers and b is not a multiple of a, the expression $\frac{b}{a}$ has no meaning; but we make to correspond to the couple a, b, a new object, different from all those we have considered hitherto, which we will denote by $\frac{b}{a}$, and which we define by the relation of equality, which satisfies our three conditions.

$$\frac{b}{a} = \frac{d}{c} .=. \quad ad = bc \qquad Df.^{50}$$

Again, the "upper limit of a class of rational numbers a" ($l'a$) was defined[51] by abstraction (b being also a class of rational numbers):

$l'a = l'b .=.$ "If x is any rational number; then, if there are any members of a greater than x, there are members of b greater than x, and *vice versa*." $Df.$

Definitions are not, strictly speaking, necessary. Thus, each proposition on irrational numbers (the foregoing "upper limits") is a proposition on aggregates of rational numbers; each proposition on rational numbers becomes a proposition on whole numbers; and so on. A definition has no need of proof, it is merely the effect of our will to represent a group of signs by a simpler expression. We have not, for example, to prove the existence of what we define. Naturally, it is proper to define existent things in practice, but

[47] Cf. papers by Vailati and De Amicis in *Riv. di Mat.*, I, 1891.
[48] *Op. cit.*, p. 47.
[49] *Vorlesungen über allgemeine Arithmetik*, Bd. I, p. 43.
[50] In this line, the sign = has, with Peano, three different meanings: the first, equality as defined by abstraction; the second, equality by Df; and the third, equality between whole numbers.
[51] Peano, *op. cit.*, p. 47; cf. *Arith. Princ.*, p. 15, and *Formulario de Mathematica*, 1905, p. 105.

UMI

sometimes we define things which do not exist. Thus Euclid,[52] in order to prove that the number of primes is infinite, said: Let us put $\delta\epsilon$ = smallest common multiple of the primes; and then proved that $\delta\epsilon$ does not exist.

We cannot define everything; to define a sign x, we must be able to compose a sign a from known signs such that we have $x=a$ D*f*. Thus, we must know some signs already. The question, Can the object x be defined? is not quite correctly put; we should rather say: Can x be defined by means of the objects $a, b, \ldots$?, and there is a certain amount of arbitrariness[53] as to which objects we take as ultimate,—the minimum of objects with which we can begin a logic, or "primitive ideas." These ideas, said Peano,[54] "must be acquired by experience or by induction; it is impossible to explain them by deduction." The primitive ideas of a science constitute the smallest dictionary which must be common to two men who speak different languages, in order that they may be able to understand one another on the subjects of this science.[55]

This determination of a primitive idea by a group of primitive propositions or postulates concerning it, was admitted under the name of "definitions by postulates" by Burali-Forti[56] as one of the three legitimate forms of definition in mathematics. To "define" an object x means: "to give one or many logical relations containing x, and such that, an element y being given, it is possible to affirm or deny the relation $x=y$."[57] In other words, x is defined

[52] Book IX, prop. 20.

[53] Thus, if by means of a, b, c, we can define d, and by means of a, b, d, we can define c, we can take for primitive ideas either a, b, c, or a, b, d. See also Russell, *op. cit.*, p. 111.

[54] *Op. cit.*, p. 50.

[55] Russell called the primitive ideas "indefinables" and enumerated them, and "logical constants" was the name he gave to all notions definable in terms of them (*op. cit.*, pp. 3, 4, 7-8, 11; Couturat, *op. cit.*, pp. 37-39).

[56] *Sur les différentes méthodes logiques pour la définitions du nombre réel*, (*Bibl. du Congrès Internat. de Phil.*, "Logique et histoire des sciences," III, pp. 294-307, especially pp. 294-296).

[57] This means (Burali-Forti, *loc. cit.*, pp. 292-293), that *every* property of x is also one of y. Certain relations which are reflexive, symmetrical, and transitive (like "is superposable on") have been denoted by =, but this was only with reference to all those properties *relative to our discourse*. Cf. also a preceding note on mathematical equality.

when one can *deduce* all the properties of x from the logical relations in question. The two other kinds of definition are (1) the "nominal definition" of x in the form $x=a$, which has already been described, and (2) the "definition by abstraction" of an operation ϕ by saying to what class a it is applicable and, x being an element of a, by establishing which are the y's of a such that $\phi y=\phi x$. This has also been described above.

* * *

Russell[58] urged against the validity of the above process of Peano's of using abstraction as a substitute for definition,[59] the fatal formal defect of not showing that only one object satisfies the definition. Thus, in the definition by abstraction of "powers" or "cardinal numbers," we consider two classes u and v which can be put in a one-one relation[60] with one another, or are equivalent. As equivalence is a reflexive, symmetrical, and transitive relation, Peano and common sense conclude that u and v have a common property, and *vice versa*; this common property we can then define as their cardinal number, so that the equality of the cardinal numbers of u and v consists in the equivalence of u and v. Instead of obtaining *one* common property[61] of similar classes, which is *the* cardinal number of the classes in question, we obtain a *class* of such properties, with no means of deciding how many terms this class contains.[62] In order to make this point clear, let us examine what is

[58] *Op. cit.*, pp. 114-115.

[59] This process of analyzing any reflexive, symmetrical, and transitive relation between the classes u and v into sameness of relation to an entity denoted by ϕu or ϕv to be obtained by abstraction, was called "definition by abstraction" by Burali-Forti in his *Logica Matematica*, published at Milan in 1894.

[60] A relation is one-one when, if x and x' have the relation in question to y, then x and x' are identical; while if x has the relation in question to y and y', then y and y' are identical. A one-one relation whose domains are u and v was denoted by Peano by $fv\phi$ placed partly between and partly after u and v (*Formulario*, 1905, p. 75).

The term "one-one" does not imply that the (as yet undefined) notion of "the number 1" is used in this definition, and such is not the case (cf. Russell, *op. cit.*, pp. 113, 305, and Couturat, *op. cit.*, pp. 31-32, 47-48).

[61] Cf. Cantor's definition by abstraction (1883) in *Zur Lehre vom Transfiniten*, pp. 23-24.

[62] Couturat (*op. cit.*, p. 48) pointed out that this class may, seemingly, be null; "a definition by abstraction shows neither the existence nor the uniqueness of the object defined."

UMI

meant in the present instance, by a common property. What is meant is, that any class has to a certain entity, its number, a relation which it has to nothing else, but which all equivalent classes (and no other entities) have to the said number. That is, there is many-one relation wihch every class has to its number and to nothing else. Thus, so far as the definition by abstraction can show, any set of entities to each of which some class has a certain many-one relation, and to one and only one of which any given class has this relation, and which are such that all classes equivalent to a given class have this relation to one and the same entity of the set, appear as the set of numbers, and any entity of this set is *the* number of some class. If then, there are many such sets of entities—and it is easy to prove that there are an infinite number of them—every class will have many numbers, and the definition wholly fails to define *the* number of a class. This argument is perfectly general, and shows that definition by abstraction is never a logically valid process.

The legitimacy of this process of Peano's requires[63] an axiom, namely that, if there is any instance of the relation in question—a transitive, symmetrical and (within its field) reflexive one between u and v—there is such a new entity as ϕu or ϕv such that our relation is analyzed into sameness of relation to the new term ϕu or ϕv. As the entity to be defined should be visible, at least to the mind's eye,[64] this axiom becomes, in the logic of relations, a proposition proved by Russell in his calculus of relations, and called by him "the principle of abstraction."[65] This principle is: "Any symmetrical and transitive relation R, of which there is at least one instance, can be expressed as the relative product of a many-one relation S and its converse, so that S subsists between each of the individuals x, y and a third term z in such a way[66] that xRy is equiv-

[63] Russell, *op. cit.*, p. 220

[64] *Ibid.*, p. 249.

[65] Russell, *op. cit.*, pp. 166-167, 116, 305; Couturat, *op. cit.*, pp. 33, 42-43, 48-50; and Russell's paper: "Sur la logique des relations," *Rev. de Math.*, VII, No. 2, § 1, Prop. 6, 2.

[66] An axiom virtually identical with this principle, but not stated with the necessary precision, or not demonstrated, is, according to Russell (*op. cit.*, p.

alent to the two propositions: $x\mathrm{S}z$ and $y\mathrm{S}z$. It is this z which is Peano's ϕx or ϕy, and is the common property of x and y;[67] and all mathematical purposes of the supposed common property are completely served when it is replaced by this z. Russell actually constructed such a z by pointing out that the requirements were satisfied by the class of terms having the given relation to a given term.

Thus, if we apply the principle of abstraction to equivalent classes, we arrive[68] at a definition of the cardinal number of u as the class of the classes similar to u.[69]

* * *

The "definition by postulates" also is not a definition.[70] An aggregate of postulates only determines the meaning of the undefined symbols to a certain extent, for the same system of postulates can be verified by many interpretations given to the undefined symbols: a system of postulates is analogous to a system of equations between many unknowns; if our postulates really determine our undefined notions uniquely, a "resolution" with respect to these unknowns results in nominal or explicit definitions. When the system of postulates contains only one primitive idea, it is easy to extract the explicit definition of the latter, for we need only say that it is "such that" it verifies the system of postulates. But it

219n), to be found in a paper by De Morgan, *Camb. Phil. Trans,*. vol. X, p. 345.

[67] The principle, then asserts "that there *are* such entities, if only we know where to look for them" (Russell, *op. cit.*, p. 249).

[68] Russell, *op. cit.*, pp. 115, 304-307.

[69] It then becomes a strictly demonstrable proposition that any class u has a cardinal number. For u itself is a member of the class called the cardinal number of u, since u is similar to itself (equivalence is a reflexive relation) and hence the cardinal is not a null class (Russell, *op. cit.*, p. 305; first given in *Rev. de Math.*, VII, p. 121).

Cf. Frege, *Die Grundlagen der Arithmetik*. . . ., Breslau, 1884, pp. 73-99.

Analogously, a nominal definition of ordinal types as a class of like relations was given by Russell (*op. cit.*, pp. 241, 313; cf. Couturat, *op. cit.*, pp. 76-77).

[70] Couturat, *op. cit.*, pp. 40-42, 57-58; Frege, "Ueber die Grundlagen der Geometrie," *Jahresber. d. deutsch. Math.-Ver.*, Bd. XII, 1903, pp. 319-324; 368-375). Cf. also Russell's criticism of Peano's way of defining finite integers, together with his proofs of Peano's primitive propositions in arithmetic, in *op. cit.*, pp. 124-128.

remains to prove the existence and uniqueness of this notion, as for every other explicit definition.

Every definition is, then, nominal;[71] the "definition by abstraction" is only necessitated by an incomplete logic which does not include a calculus of relations, while the introduction of primitive ideas other than those of logic into arithmetic can, as Russell[72] has shown, be avoided.

* * *

Hilbert later applied the same axiomatic method to the principles of arithmetic, and exposed his results at some length in a lecture *Ueber die Grundlagen der Logik und der Arithmetik* (*Verh. des. 3. internat. Math.-Kongresses in Heidelberg im August, 1904*, pp. 174-185, Leipsic, 1905; translated in *The Monist*, July, 1905, Vol. XV, pp. 338-352). At the beginning of this, he announced complacently that "to-day in researches on the foundations of geometry we are essentially agreed as to the procedures to be adopted." If the procedure is the procedure of Hilbert, in which the essential factor of existence of the object supposed to be defined by the axioms is disregarded, and consequently in which one cannot be sure that one is arguing about anything at all, this is most certainly not the case; in America, for instance, there is the important work on geometry of O. Veblen, who gives chains of axioms for various geometries, but proves the existence-theorems.

Hilbert's view is that there is an essential difference between an examination of the foundations of geometry and one of the foundations of arithmetic, because, in the former case, the mutual compatibility of the axioms can be proved by *arithmetical* constructions, while in the latter case, this is naturally impossible, and "in the founding of arithmetic, the appeal to another basal science seems unallowable." But (cf. on this point M. Pieri, *Sur la compatibilité des axiomes de l'arithmétique, Revue de métaphys. et de morale*, March, 1906, XIV, pp. 196-207) the basal science for arithmetic can be, as Russell's whole work has shown, logic—including the

[71] Russell, *op. cit.*, p. 112; Couturat, *op. cit.*, p. 43.

[72] *Op. cit.*, pp. 8-9, 497-498.

logic of relations, and logic alone is sufficient for the definition of *all* the conceptions of pure mathematics.

What is Hilbert's difficulty in the founding of arithmetic on logic appears from his criticism of Frege—a criticism which applies also to the earlier work of Russell. Hilbert quite correctly observes that "inasmuch as he (Frege), true to his plan, takes....as axiom, that a concept (an aggregate) is defined and immediately available, provided only it be determined for every object, whether it falls under the concept or not, and also in doing this subjects the concept "every" to no restriction [cf. also Cantor, *Math. Ann.*, Bd. XX, 1882, pp. 114-115], he exposes himself to just those paradoxes of the theory of aggregates, which lie, for instance, in the concept of the aggregate of all aggregates [cf. below], and which, it seems to me, show that the conceptions and means of investigation of logic, taken in the usual sense, are not adequate to the rigorous requirements set up by the theory of aggregates." Hilbert's aim, from the very outset, was to avoid such contradictions.

However, though Hilbert develops the conception of the various finite and transfinite numbers in order, and, at each stage, restricts the word *all* to apply only to those entities already introduced, and, by this method, which he has not been the only one to adopt, never gets to Burali-Forti's contradiction; yet he does not seem to me to avoid Russell's contradiction, since "non-existent" means, with him, non-entity, and consequently his "class" of the existent is "the class" of all things.

We will not, in this short account, attempt a detailed criticism of Hilbert's lecture; and will merely remark that the creation by the mind of various "thought-things" governed by certain axioms is, even if such creation is possible, at least unnecessary, for another way, which Frege and Russell had previously followed, is preferable if for no other reason than that Occam's principle is observed (cf. below). As our present object is solely the discrediting of "definitions by postulates," we may merely refer, for other criticisms, to Couturat, *Rev. de métaphys. et de morale*, March, 1906, XIV, pp. 234-235; and Pieri, *ibid.*, p. 200.

When we define a class[73] we must, in order to be able to reason on this class and investigate its properties, prove that there is at least one member of this class; in other words, that the class is not null, or "exists,"[74] so that the conditions which define it are not logically incompatible. Every definition must, then, be accompanied by an *existence-theorem* (or postulate),[75] and, if we have to speak of "the" member of a class, we must prove that, if two individuals are members of the class in question, they are identical.

v.

We must now consider the notion of equality (=) in logic and in mathematics. In mathematics the process is frequently adopted of defining equality for, say, whole numbers,[76] and then *redefining* equality for other classes of numbers, such as ratios and real numbers. If, as is usually the case, the same sign (=) is used for these different conceptions of equality, there may be confusion; but, altogether apart from this question, which merely concerns the symbols used, there is a real question of principle involved, which makes this redefinition of equality objectionable: the new meanings of equality imply, in fact, a lack of thoroughness in the analysis of these meanings, which always involve the *identity* (the original meaning of equality in logic) of the "equal" objects *in some respect.*

The meaning of "equality" in logic is identity; when we say there $a=b$ we mean that a and b are different names for the same thing;[77] or, in formal language, *every* property of the thing denoted

[73] Couturat (*op. cit.*, p. 39) stated that the term defined is *always* a class; Russell (*op. cit.*, pp. 63, 497) did not go as far as this, and it may be remarked that some of the different kinds of "number" defined in analysis are relations.

[74] Russell, *op. cit.*, pp. 21, 32; Couturat, *op. cit.*, pp. 25-26.

[75] Cf. Russell, *op. cit.*, pp. ix, 322; Couturat, *op. cit.*, pp. 39-40. Russell (*op. cit.*, p. 497) sketched the chain of proofs that the numbers and other classes defined in mathematics exist.

[76] Thus we may define the members of two classes u and v to be "equal," when there can be set up a one-one correspondence between the members of u and those of v.

The sign = is to be distinguished from "= D*f*."

[77] See Dedekind, *Was sind and was sollen die Zahlen?* 1887 and 1893, pp. 1-2 (translation in Dedekind's *Essays on Number*, Chicago, pp. 44-45); Schröder, *op. cit.*, Bd. I, Leipsic, 1890, pp. 184-186; Peano, *Formulaire de mathématiques*, t. II, § 1, prop. 80; Burali-Forti, *Bibl. du Congrès Internat. de Phil.*, t. III, p. 292; Frege, *Grundgesetze der Arithmetik*, Bd. I, Jena, 1893, p. ix, and II, 1902, p. 71.

by a is also one of the things denoted by b.[78] Now the notion of equality, so often used in mathematics, in which not *every* property of a is one of b, but the relation connecting a and b is, like equality, reflexive, symmetrical, and transitive, is always one of the equality as defined above of certain *functions* of a and b. Thus, some geometricians have extended the meaning of equality and have called a (plane and rectilinear) figure a "equal" to a figure b when a is superposable (after dissection, if necessary) on b; this relation of superposability is reflexive, symmetrical, and transitive, and this relation which we may write aSb, can be put into the form $\phi(a)=\phi(b)$ by letting "$\phi(\ )$" stand for "the area of ( )." Similarly, the relation of parallelism (analogous, in many ways, to equality) between two straight lines a and b transforms into an *identity* between certain functions of $=$ the directions of a and b.[79]

It is better to avoid introducing new conceptions unless they are really necessary, and new conceptions of equality are not necessary, and have the disadvantage, further, of rendering confusion possible. The decisive factor is, here as in the question as to whether numbers and other mathematical conceptions are to be defined logically or to be regarded as entities created by our minds, that *entia non sunt multiplicanda praeter necessitatem*, and hence that the problem of first principles is a minimal problem.

VI.

Most mathematicians would say that "existence" is absence of contradiction; whereas we have defined logical (or, what is the same thing, mathematical) existence as an attribute applying to a *class* a which is not null.[80] The proof that a class "exists" or is not

[78] Supposing that properties (propositional functions) determine *classes* in the manner already explained, this may be also put in the form: "$a=b$" means that *every* class which contains the object a also contains b. On Frege's theory that equality is not an identity of names, but expresses an identity of what he calls "denotation" (*Bedeutung*) together with a diversity of "signification" (*Sinn*), see his essay "Ueber Sinn und Bedeutung" in *Zeitsch. für Phil.*, C, 1892. We shall return to this point.

[79] Cf. Frege, *Grundlagen*, pp. 76-77; Couturat, *op. cit.*, p. 49, note; Burali-Forti, "Sur l'égalité, et sur l'introduction des éléments dérivés dans la science," *Enseignement math.*, I, 1899, pp. 246-261, and the above-mentioned *Congrès* paper (pp. 289-307).

[80] Cf. Russell, *Mind*, N. S., XIV, 1905, p. 398.

null, is always brought about by the actual construction or indication of an individual belonging to the class, and to inquire if an individual "exists" has no meaning.[81] This, now, is the point: mathematicians require a proof of the "existence" of an individual, logicians reply that "existence is a property of classes alone.[82] And the logicians' reply is obviously not satisfactory:[83] it leads one to suspect that there may be individuals, which may be used to prove the existence of classes to which they belong, and which are self-contradictory.

Let us examine a case in which mathematicians have proved what they would call the non-existence of an individual, namely, the self-contradictory nature of a complex number with more than two independent unities which satisfies all the formal laws of ordinary algebra. But we may also express the result of this as: the class of such complex numbers is null, or non-existent, and such a number is not an entity at all. Mathematicians, in fact, have used the word "exists" in two senses: (1) A class exists when we can find a member of it;[84] and (2) an individual does not exist when it is self-contradictory. Logicians use "exists" in the first sense only, for the second sense is merely: a class of such individuals does not "exist," in the first sense. The question is merely a verbal one, the mathematician's usage is confusing, the logician's is not.[85]

What is of great importance in this connection is that, as we shall see in the second part, while we may define the null-class as $x \ni \phi x$, where ϕx is false for every entity x (such as $\phi x =$ "x is not identical with x"), we may also have a non-entity, which may be

[81] Of course, a class (even a null, or non-existent *class*) can be considered as an individual with respect to a class of classes.

[82] Cf. Couturat, *Rev. de métaphys. et de morale*, March, 1906, pp. 232-234; Poincaré, *ibid.*, September, 1906.

[83] Couturat's comments are not always accurately expressed: "On ne démontre pas l'existence d'un individu comme tel. Les individus, par cela même qu'ils sont des individus, sont toujours considerés comme existants; ou plutôt la question ne se pose pas pour eux." To talk of an *existent* individual (even though the epithet obviously was not meant to be taken literally) only increases confusion; the fact is that an entity, if it *is* an entity, is a self-contradictory entity (an entity which is a non-entity).

[84] Cf., for example, Dedekind, *Essays on Number*, 45, 49, 58.

[85] That "exists" in mathematics often means "has being" or "is an entity" is one of the discoveries whose genesis will be described in a later issue.

proved to be a member of the null-class, determined by $x \ni \psi x$, where ψx is *not* false for every entity x. Thus ψx="x is not a member of x" is true for some (if not all) x's, but $x \ni \psi x$ is not an entity. And further, this $x \ni \psi x$ appears to be an existent class; a strange dilemma for those who rely on intuition.

As early as 1884, Frege,[86] when criticising Hankel's formalist theory of the "numbers" of analysis, gave, in an important passage, the modern logical view of existence, including the remark, that a contradictory concept is permissible—but has no extension,[87]—and that the process[88] of introducing new "signs" as numbers, conformably to the "principle of permanence," is an error.

The chain of the existence-theorems for cardinal and ordinal arithmetic is, now, as follows:[89]

It may be shown, to begin with, that no definite class embraces all terms: this results from the fact that, since 0 is a cardinal number, the number of numbers up to and including a finite number n is $n+1$. Further, if n be a finite number, $n+1$ is a new finite number different from all its predecessors. Hence finite cardinals form a "progression,"[90] and therefore the ordinal number ω and the cardinal number $\aleph_0$ exist (in the mathematical sense). Hence, by mere rearrangements of the series of finite cardinal numbers, we obtain all ordinal numbers of Cantor's second class. We may now define ω_1 as the class of serial relations such that, if u be a class contained in the field of one of them, to say that u has successors implies and is implied by saying that u has a finite number of, or $\aleph_0$, terms; and it is easy to show that the series of ordinal numbers of the first and second classes in order of magnitude is of this type. Hence the

[86] *Grundlagen*, pp. 105-106, 107-108.

[87] This may be illustrated by Euclid's "$\delta\epsilon$" (see above).

[88] This process is used in the formal theory, but there is no doubt that Cantor did not, in spite of a statement of Pringsheim's *Encykl. der math. Wiss.* (Bd. I, p. 69) consider (at least in his later works) his transfinite numbers as a generalization of the *a priori* given concept of finite number. Also Schönflies's use of the "principle of permanence" to obtain the concepts of infinite numbers and types (*Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Leipsic, 1900, pp. 3-4, 27) must be regarded as a mistake.

[89] *Op. cit.*, pp. 322-323; cf. pp. ix, 111-116, 277-281, 313, 321-322, 497-498; and *Hibbert Journal*, July, 1904, pp. 810-811.

[90] *Op. cit.*, p. 239.

existence of ω_1 is proved; and $\aleph_1$ is defined to be the cardinal number of terms in a series whose generating relation is of the type ω_1. Hence, we can advance to ω_2 and $\aleph_2$, to ω_ω, and $\aleph_\omega$, and so on. This process gives us a one-one correlation of ordinals with (some) cardinals: it is evident that, by extending the process, we can make each cardinal which can belong to a well-ordered series correspond to one and only one ordinal. Cantor assumed that every class is the field of some well-ordered series, and hence deduced that all cardinals are Alephs.[91] This assumption seemed to Russell unwarranted.

VII.

Another, and rather different, example of the use of this "principle of abstraction" was given by Russell[92] in his definition of a real number. A real number was, as we have seen above, defined by Peano by abstraction: but Russell gave a nominal definition of a real number as a class, which can be proved to be an" existent" class,[93] and which has all the mathematical properties commonly assigned to a real number.

Any class of rational numbers[94] which is not null, which does not comprise all rational numbers,[95] and which comprises all those less than any one of its elements, is called a *segment* of rationals. To each rational r belongs one segment (of rationals less than it),

[91] See below.

[92] *Op. cit.*, pp. 270-286; Couturat, *op. cit.*, pp. 85-89.

[93] It is only by defining a number nominally, and as a class that its "existence" can be proved.

[94] The rational numbers here used are *signless* ratios or *relations* of finite cardinal numbers (see Russell, *op. cit.*, pp. 149-150; Couturat, *op. cit.*, pp. 79-81), or again they may be defined as Frege (*Grundlagen*, pp. 114-115) seemed to have urged, as classes (which can be proved to "exist") of couples. In either case, these rationals must be carefully distinguished from the rationals *with sign* (positive and negative), in the same way that a cardinal number n (a class) is not to be confused with the "positive integer" $+n$ (a relation, see Russell, *op. cit.*, p. 229; Couturat, *op. cit.*, pp. 80, 89). Cf. the distinction carried out between integers, integers with sign, rationals, rationals with sign, and so on, by Peano, with only minor mistakes in his *Formulario de mathematica*, V, 1905, pp. 83, 95-100, etc.

[95] By dropping the first condition alone, we may introduce zero, and, by dropping the second condition alone, we may introduce infinity, as limiting cases of segments (Russell, *op. cit.*, pp. 273-274; Couturat, *op. cit.*, p. 89; cf. Jourdain, *Journ. für Math.*, Bd. CXXVIII, 1905, p. 186).

but not inversely;[96] and, indeed, the class of segments is not capable of a one-one correlation with the class of rationals.

If, now, we confine our attention to those segments which have no rational maximum, or, in other words, the segments (u) such that every term of u is less than some other term of u, and consider those other classes v of rationals such that, if x is any member of u, there is a member of v greater than it, and, if y is any member of v, there is a member of u greater than it. This relation of v to u may be expressed:[97] "v is coherent to u"; and Cantor[98] considered this relation of being coherent (*zusammengehörig*) when, as is sufficient when u is any segment of rationals (of type η), v is a "fundamental" series (of type ω)

$$\omega_1, \omega_2, \ldots, \omega_\nu, \ldots;$$

while the series cannot be of *finite* type if u has no maximum. Another class w (arranged in type $\omega \cdot 2$, for example, or again in type ω), may also be coherent to u, and the relation of being coherent may be proved to be symmetrical and transitive. From this we infer that both v and w have to some third term (the "common property") a relation which neither has to any other term; and this third term may be taken to be the segment u which both define, and thus u is said to be the real number which all classes coherent to u define.[99]

Now there is a difference between the use of the "principle of abstraction" here and its use in defining the cardinal number of a class. Here a class of rationals has the relation of "being coherent to" its real number, there a class had the relation ϵ to its cardinal number. And we may frame a definition of a real number like, in this respect, that of a cardinal number. If a is a class (finite or infinite) of rationals, we may define the real number be-

[96] This fact may be described by saying that there are irrational segments.

[97] Russell, *op. cit.*, p. 274.

[98] *Math. Ann.*, Bd. XLVI, 1895, p. 508.

[99] Peano (*Formulario*, 1905, p. 10) defined a real number $l'u$, the "upper limit of u," where u is a class of rationals, by the abstraction:

$$l'u = l'v \,.=.\, \eta u = \eta v \qquad Df.,$$

where ηu and ηv are segments. Russell's definition is:

$$l'u = \eta u,$$

which obviously satisfies the above equation, and does not require any new meaning of $=$ besides logical identity (cf. above § 5).

longing to u ($l'u$) as "the class of all those classes which are coherent to u."

We shall find an analogue to these two definitions of real number in the definitions of cardinal and ordinal number which I propose (in what I call my "second theory"), as seeming, for a certain reason, preferable to Russell's though the new definitions are not essential to the theory.[100] With Russell's the relation of a class α to its cardinal number is ϵ, with mine this relation is "is similar to"; and both definitions satisfy the necessary requirements of being nominal and not requiring the re-definition of $=$, which has thus the same meaning (of identity) throughout all logic and mathematics.

[TO BE CONTINUED.]

PHILIP E. B. JOURDAIN.

BROADWINDSOR, BEAMINSTER, DORSET, ENGLAND.

[100] This "second theory," in which cardinal numbers are defined, by an extended induction, to be classes of the preceding cardinal numbers, seems necessary if we are to avoid—what we must in what will be referred to as the "limitation-of-size theory"—defining a number as a class equivalent to the class of all classes. However, it must be acknowledged that Russell, by his "no-classes theory," has made such an attempt to improve the older theory superfluous.

CRITICISMS AND DISCUSSIONS.

ODDLY-EVEN MAGIC SQUARES.*

A convenient classification of magic squares is found by recognition of the root as either a prime number or evenly-even, or oddly-even, or oddly-odd. These four classes have many common traits, but owing to some characteristic differences, a universal rule of construction has hitherto seemed unattainable. The oddly-even squares especially, have proved intractable to methods that are readily applicable to the other classes, and it is commonly believed that they are incapable of attaining the high degree of magical character which appears in those others.

Mr. W. S. Andrews, in *Magic Squares and Cubes,* page 183, has indeed presented a remarkable composite 10-square, a quasi 5-square, formed of twenty-five quadrate groups of consecutive terms, that is, a series of progressive 2-squares. That specimen has most of the diagonals correct, and the author ventures a prophecy that the number of such diagonals may in some way be increased. As some extensive explorations, recently made along those lines, have reached a very high latitude, the results will now be presented, showing a plan for giving to this peculiar sort, more than the ordinary magical properties.

Problem: To make oddly-even squares which shall have proper summation in all diagonal and rectangular rows except two, which two shall contain S—1 and S+1 respectively. This problem is solved by the use of auxiliary squares.

If n is an oddly-even root, and the natural series 1, 2, 3 etc. to n^2 is written in current groups of four terms, thus:

1.2.3.4.—5.6.7.8.—9.10.11.12.—13.14.15.16. etc.
0.1.2.3.—0.1.2.3.—0. 1. 2. 3.— 0. 1. 2. 3. etc.

1 5 9 13 etc.

* Diagrams drawn by W. S. Andrews, Schenectady, N. Y.

then from each current group a series 0.1.2.3 may be subtracted, leaving a series 1.5.9.13 etc. to n^2—3, a regular progression of $n^2/4$ terms available for constructing a square whose side is $n/2$. As there are four such series, four such squares, exactly alike, readily made magic by well-known rules, when fitted together around a center, will constitute an oddly-even square possessing the magical character to a high degree. This will serve as the principal auxiliary. Another square of the same size must now be filled with the series 0.1.2.3 repeated $n^2/4$ times. The summation $3n/2$ being always odd, cannot be secured at once in every line, nor equally divided in the half lines, but all diagonal and all rectangular rows, except two of the latter, can be made to sum up correctly. Hence the completed square will show a minimum of imperfection.

In illustration of these general principles, a few examples will be given, beginning properly with the 2-square, smallest of all and first of the oddly-even. This is but an embryo, yet it exhibits in its nucleated cells some germs of the magical character, capable of indefinite expansion and growth, not only in connection with those of its own sort, but also with all the other sorts. Everything being reduced to lowest terms, a very general, if not a universal principle of construction may be discovered here. Proceeding strictly by rule, the series 1.2.3.4. affords only the term 1. repeated four times, and the series 0.1.2.3. taken once. The main auxiliary (Fig. 1) is a genuine quartered 2-square, equal and identical and regular and continuous every way. S=2.

1	1
1	1

Fig. 1.

0	1
2	3

Fig. 2

0	2
3	1

Fig. 3.

0	3
2	1

Fig. 4.

1	2
3	4

Fig. 5.

The second auxiliary (Fig. 2) taking the terms in direct order, has eight lines of summation, showing equality, S=3, in all four diagonals, while the four rectangular rows give inequalities 1.5 and 2.4; an exact balance of values. This second auxiliary may pass through eight reversed, inverted or revolved phases, its semi-magic character being unchanged. Other orders may be employed, as shown in Figs. 3 and 4, bringing equality into horizontal or vertical rows, but not in both directions at the same time. Now any one of these variables may combine with the constant shown in Fig. 1, developing as many as twenty-four different arrangements of the 2-square, one example of which is given in Fig. 5.

It can not become magic unless all its terms are equal; a series whose common difference is reduced to zero. As already suggested, this 2-square plays an important part in the present scheme for producing larger squares, pervading them with its kaleidoscopic changes, and forming, we may say, the very warp and woof of their substance and structure.

The 6-square now claims particular attention. The main auxiliary, Fig. 6, consists of four 3-squares, each containing the series 1.5.9.13 etc. to 33. The 3-square is infantile; it has but one plan of construction; it is indeed regular and can not be otherwise, but it is imperfect. However, in this combination each of the four has a different aspect, reversed or inverted so that the inequalities of partial diagonals exactly balance. With this adjustment of subsquares the 6-square as a whole becomes a perfect quartered square, S=102; it is a quasi 2-square analogous to Fig. 1.

13	33	5	5	33	13
9	17	25	25	17	9
29	1	21	21	1	29
29	1	21	21	1	29
9	17	25	25	17	9
13	33	5	5	33	13

Fig. 6.

0	2	2	0	3	2
3	1	1	3	0	1
0	2	2	0	3	2
3	0	1	3	1	1
0	3	2	0	2	2
3	0	1	3	1	1

Fig. 7.

The four initial terms, 1.1.1.1 symmetrically placed, are now to be regarded as one group, a 2-square scattered into the four quarters; so also with the other groups 5.5.5.5 etc. Lines connecting like terms in each quarter will form squares or other rectangles, a pattern, as shown in Figure 9, with which the second auxiliary must agree. The series 0.1.2.3 is used nine times to form this second square as in Figure 7. There are two conditions: to secure in as many lines as possible the proper summation, and also an adjustment to the pattern of Fig. 6. For in order that the square which is to be produced by combination of the two auxiliaries shall contain all the terms of the original series, 1 to 36, a group 0.1.2.3 of the one must correspond with the group 1.1.1.1 of the other, so as to restore by addition the first current group 1.2.3.4. Another set 0.1.2.3 must coincide with the 5.5.5.5; another with the 9.9.9.9 and so on with all the groups. The auxiliary Fig. 7 meets these conditions. It has all diagonals cor-

UMI

rect, and also all rectangular rows, except the 2d and 5th verticals, which sum up respectively 8 and 10.

Consequently, the finished square Fig. 8 shows inequality in the corresponding rows. However, the original series has been restored, the current groups scattered according to the pattern, and although not strictly magic it has the inevitable inequality reduced to a minimum. The faulty verticals can be easily equalized by transposing the 33 and 34 or some other pair of numbers therein, but the four diagonals that pass through the pair will then become incorrect, and however these inequalities may be shifted about they can never be wholly eliminated. It is obvious that many varieties of the finished square having the same properties may be obtained by reversing or revolving either of the auxiliaries, and many more by some other arrangement of the subsquares. It will be observed that in Fig. 6 the group 21 is at the center, and that each 3-square may revolve on its main diagonal, 1 and 25, 9 and 33, 29 and 5 changing places. Now the subsquares may be placed so as to bring

13	35	7	5	36	15
12	18	26	28	17	10
29	3	23	21	4	31
32	1	22	24	2	30
9	20	27	25	19	11
16	33	6	8	34	14

Fig. 8.

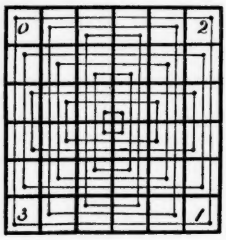


Fig. 9.

either the 5 or the 13 or the 29 group at the center, with two changes in each case. So that there may be $8\times8\times8=512$ variations of this kind. There are other possible arrangements of the subsquares that will preserve the balance of the partial diagonals, but the pattern will be partly rhomboidal and the concentric figures tilted to right and left. These will require special adaptation of the second auxiliary.

We come now to the 10-square, no longer hampered as in the 6-square, by the imperfection of the subsquares. The main auxiliary Fig. 10 consists of four 5-squares, precisely alike, each containing the series 1.5.9 etc. to 97, S=245, in every respect regular and continuous. All four face the same way, but they might have been written right and left, as was *necessary* for the 3-square. The

groups 1.1.1.1, 5.5.5.5 etc. are analogously located, and the pattern consists of equal squares, not concentric but overlapping. The 10-square as a whole is regular and continuous. S=490.

73	29	85	41	17	73	29	85	41	17
45	1	77	33	89	45	1	77	33	89
37	93	49	5	61	37	93	49	5	61
9	65	21	97	53	9	65	21	97	53
81	57	13	69	25	81	57	13	69	25
73	29	85	41	17	73	29	85	41	17
45	1	77	33	89	45	1	77	33	89
37	93	49	5	61	37	93	49	5	61
9	65	21	97	53	9	65	21	97	53
81	57	13	69	25	81	57	13	69	25

Fig. 10.

The second auxiliary Fig. 11 is supposed to have at first the normal arrangement in the top line 0.3.0.0.3.2.2.1.2.2. which would lead to correct results in the rectangular rows, but an alternation of values in all diagonals, 14 or 16. This has been equalized

0	3	1	0	3	2	2	0	2	2
3	0	2	3	0	1	1	3	1	1
0	3	1	0	3	2	2	0	2	2
3	0	2	3	0	1	1	3	1	1
0	3	1	0	3	2	2	0	2	2
3	0	3	3	0	1	1	2	1	1
0	3	0	0	3	2	2	1	2	2
3	0	3	3	0	1	1	2	1	1
0	3	0	0	3	2	2	1	2	2
3	0	3	3	0	1	1	2	1	1

Fig. 11.

by exchange of half the middle columns, right and left, making all the diagonals = 15, but as the portions exchanged are unequal those two columns are unbalanced. The exchange of half columns might have taken place in the 1st and 8th, or in the 2d and 6th, either

the upper or the lower half, or otherwise symmetrically, the same results following.

The resultant square Fig. 12 contains all the original series, 1 to 100; it has the constant S=505 in thirty-eight out of the total of forty rows. When made magic by transposition of 15 and 16, or some other pair of numbers in those affected columns, the four diagonals that pass through such pair must bear the inequality.

73	32	86	41	20	75	31	85	43	19
48	1	79	36	89	46	2	80	34	90
37	96	50	5	64	39	95	49	7	63
12	65	23	100	53	10	66	24	98	54
81	60	14	69	28	83	59	13	71	27
76	29	88	44	17	74	30	87	42	18
45	4	77	33	92	47	3	78	35	91
40	93	52	8	61	38	94	51	6	62
9	68	21	97	56	11	67	22	99	55
84	57	16	72	25	82	58	15	70	26

Fig. 12.

Here, as in the previous example, the object is to give the second auxiliary equal summation in all diagonals at the expense of two verticals, and then to correct the corresponding error of the finished square by exchange of two numbers that differ by unity.

In all cases the main auxiliary is a quartered square, but the second auxiliary is not; hence the completed square cannot have the half lines equal, since S is always an odd number. However, there are some remarkable combinations and progressions. For instance in Fig. 12 the half lines in the top row are 252+253; in the second row 253+252; and so on, alternating all the way down. Also in the top row the alternate numbers 73+86+20+31+43=253 and the 32, 41 etc. of course = 252. The same peculiarity is found in all the rows. Figs. 10 and 11 have similar combinations. Also Figs. 6, 7 and 8. This gives rise to some Nasik progressions. Thus in Fig. 10 from upper left corner by an oblique step one cell to the right and five cells down: 73+29+85+41 etc. ten terms, practically the same as the top row = 490. This progression may be taken right or left, up or down, starting from any cell at pleasure. In Fig. 11 the ten terms will always give the constant S=15 by the

knight's move 2:1 or 1:2 or by the elongated step 3:1. Fig. 12 has not so much of the Nasik property. The oblique step one to the right and five down, 73+29+86+44 etc. ten terms = 505. This progression may start from any cell moving up and down, right and left by a sort of zigzag. The second auxiliary is richest in this Nasik property, the main auxiliary less so, as it is made by the knight's move; and the completed square still less so, as the other two neutralize each other to some extent. A vast number of variations may be obtained in the larger squares, as the subsquares admit of so many different constructive plans.

The examples already presented may serve as models for the larger sizes; these are familiar and easily handled, and they clearly show the rationale of the process. If any one wishes to traverse wider areas and to set down more numbers in rank and file, no further computations are required. The terms 0.1.2.3 are always employed: the series 1.5.9 etc. to 97, and after that 101.105.109 and so on. The principal auxiliary may be made magic by any approved process as elegant and elaborate as desired, the four subsquares being facsimiles. The second auxiliary has for all sizes an arrangement analogous to that already given which may be tabulated as follows:

		top row
6-square,	0 3 0 — 2 2 2	
10-square,	0 3 0 0 3 — 2 2 1 2 2	" "
14-square,	0 3 3 0 0 0 3 — 2 2 2 1 2 2 1	" "
18-square,	0 3 3 3 0 0 0 0 3 — 2 2 2 2 1 2 1 1 2	" "

etc.

The top row being thus written, under each term is placed its complement, and all succeeding rows follow the same rule, so that the 1st, 3d, 5th etc. are the same, and the 4th, 6th, 8th etc. are repetitions of the 2d. This brings all the 0.3 terms on one side and all the 1.2 terms on the opposite. In columns there is a regular alternation of like terms; in horizontals the like terms are mostly consecutive, thus bringing the diagonals more nearly to an equality so that they may be corrected by wholesale at one operation. This systematic and somewhat mechanical arrangement insures correct summation in rows and columns, facilitates the handling of diagonals, and provides automatically for the required pattern of the 2-squares, in which both the auxiliaries and the completed square must agree. In making a square from the table it should be observed that an exchange of half columns is required, either the

upper or the lower half, preferably of the middle columns; but as we have seen in the 10-square, several other points may be found suitable for the exchange.

This plan and process for developing to so high a degree of excellence, the oddly-even squares, starting with the 2-square, and constantly employing its endless combinations, is equally applicable to the evenly-even squares. They do not need it, as there are many well-known, convenient and expeditious methods for their construction. However, in closing we will give a specimen of the 4-square, type of all that class, showing the pervading influence therein of the truly ubiquitous 2-square.

1	5	13	9
13	9	1	5
1	5	13	9
13	9	1	5

Fig. 13.

0	3	0	3
1	2	1	2
3	0	3	0
2	1	2	1

Fig. 14.

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

Fig. 15.

The primaries Figs. 13 and 14 as well as the complete square Fig. 15, singly and together fill the bill with no discount. Each is a quartered square, magic to a high degree. Each contains numerous 2-squares, four being compact in the quarters and five others overlapping. And there are many more variously scattered abroad especially in Fig. 14. While these specimens seem to conform exactly to foregoing rules they were actually made by continuous process using the knight's move 2:1 and 1:2. The pattern is rhomboidal.

In all the combinations here presented, and especially in these last specimens, the 2-square is pervasive and organic. "So we have a symmetry," as one of our philosophical writers has said—"which is astonishing, and might be deemed magical, if it were not a matter of intrinsic necessity."

D. F. SAVAGE.

NOTES ON ODDLY-EVEN MAGIC SQUARES.

The article on oddly-even squares by Mr. D. F. Savage in the current number of *The Monist* is a valuable contribution to the general literature on magic squares. Mr. Savage has not only clearly

described a clever and unique method of constructing oddly-even squares, but he has also lucidly demonstrated the apparent limit of their possible perfection.

The arrangement of concentric quartets of four consecutive numbers in his 6×6 square is strikingly peculiar, and in studying this feature it occurred to the writer that it might be employed in the development of these squares by a direct and continuous process, using the arithmetical series 1 to n^2 taken in groups of four consecutive terms, 1.2.3.4. —5.6.7.8. etc.

The constructive method used by Mr. Savage is based on the well-known and elegant plan of De la Hire, but the two number series which he has chosen for the first and second auxiliary squares are unusual, if not entirely new. It is difficult to see how these unique squares could have been originally evolved by any other method than that adopted by Mr. Savage, and the different constructive scheme presented herewith must be regarded as only a natural outcome of the study of his original plan. It may also tend to throw a little additional light on the "ubiquitous 2×2 square" and to make somewhat clearer the peculiar features that obtain in these oddly-even squares.

Referring to Fig. 1 (which is a reflected inversion of Fig. 7 in the article by Mr. Savage and therefore requires no further explanation) it will be seen that this square contains nine quadrate groups of the series 0.1.2.3., the numbers in each group being

3	0	1	3	1	1
0	3	2	0	2	2
3	0	1	3	1	1
0	2	2	0	3	2
3	1	1	3	0	1
0	2	2	0	3	2

Fig. 1.

scattered in each of the 3×3 quarters, and in concentric relationship to the 6×6 square. The numbers of these quadrate groups are not, however, distributed in any apparent order as viewed numerically, although the diagram of their constructive forms, which will be referred to later on, reveals the symmetry of their arrangement.

Any middle outside cell of the 3×3 quarters containing a

cypher can be used as a starting point for a 6×6 square, and inspection will show four such cells in Fig. 1.

Selecting the second cell from the left in the upper line to start from, the numbers in the quadrate concentric group of which this cell is a member will be seen to have the formation shown in Fig.

Fig. 2.

2A, so the first group of four numbers (1.2.3.4) in the series 1 to 36 are similarly placed in Fig. 3, running also in the same *relative* numerical order.

To secure magic results in the completed square, each succeeding entry in the 3×3 quarters must follow the last entry in *magic square order*. For the next entry in Fig. 3 there is consequently a choice of two cells. Selecting the lower right-hand corner cell of the 3×3 quarter of Fig. 1 used at the start, it is seen to be occupied by 1, and the formation of the quadrate concentric

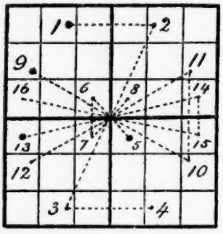


Fig. 3.

32	1	22	24	2	30
9	20	27	25	19	11
16	33	6	8	34	14
13	35	7	5	36	15
12	18	26	28	17	10
29	3	23	21	4	31

Fig. 4.

group is as shown in Fig. 2B. The terms 5.6.7.8. are therefore entered in Fig. 3 in similarly located cells, and as before, in the same relative numerical order. The next quadrate group of 9.10. 11.12 have the order shown in Fig. 2C,—13.14.15.16 are arranged as in Fig. 2D, and so on until all of the 36 cells are filled. The resulting finished square is shown in Fig. 4.

Fig. 5 shows the different forms of the nine consecutive quadrate groups contained in Fig. 4, written in regular order, and it discloses the harmonious relationship of the couplets.

There are two alternative forms for the first group, as shown in Fig. 5. If the square is to be pan-diagonal or continuous at the

expense of the summation of two vertical columns, the right-hand form must be used, but if the square is to be strictly magic at the expense of making four diagonals incorrect, then the left-hand form is correct.

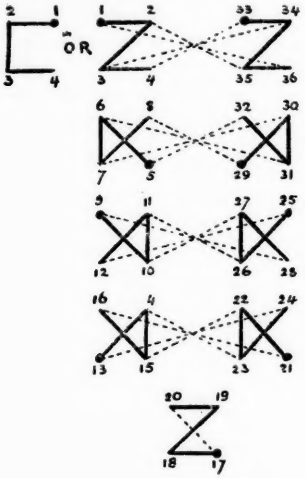


Fig. 5.

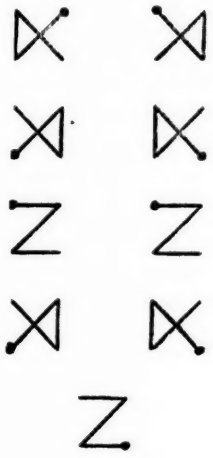


Fig. 6.

This graphic presentation of number order is instructive, as it shows at a glance certain structural peculiarities which are not apparent on the face of the square.

8	25	22	24	26	6
33	20	3	1	19	35
16	9	30	32	10	14
13	11	31	29	12	15
36	18	2	4	17	34
5	27	23	21	28	7

Fig. 7.

Another of the many variants of this 6×6 square may be made by starting from the fourth cell of the second line in Fig. 1, this being also a middle outside cell of a 3×3 square.

Under this change the forms of the quadrate groups are shown in Fig. 6, the resulting square being given in Fig. 7.

When these 6×6 squares are made pan-diagonal, i. e., perfect in all their diagonals, the normal couplets are arranged in harmonic relation throughout the square, the two paired numbers that equal n^2+1 being always located in the same diagonal and equally spaced $n/2$ cells apart. If the square is made strictly magic, however, this harmonic arrangement of the couplets is naturally disturbed in the imperfect diagonals.

The above remarks and rules will of course apply generally to 10×10 and larger squares of this class. A 10×10 square modified

20	41	86	32	73	75	31	85	43	19
89	36	79	1	48	46	2	80	34	90
63	5	50	93	40	39	95	49	7	64
56	97	23	68	9	10	66	24	98	54
25	72	14	59	84	83	57	13	71	27
26	70	16	58	82	81	60	15	69	28
55	99	21	67	11	12	65	22	100	53
62	6	52	94	37	38	96	51	8	61
91	35	77	3	47	45	4	78	33	92
18	44	88	30	74	76	29	87	42	17

+1 −1

Fig. 8.

from Mr. Savage's example to secure the harmonic arrangement of the couplets, as above referred to, is given in Fig. 8.

W. S. ANDREWS.

SCHENECTADY, N. Y.

BOLYAI, LOBATCHEVSKY, RUSSELL.

In his book on the *Foundations of Mathematics,* Dr. Carus wonders what sort of pistols Bolyai used in the famous duels where, like Ivanhoe, he discomfited the whole series of challengers. But Bolyai was a compound of Saladin and Richard, fighting with a Damascus blade which cut silken cushions or chopped iron. Franz Schmidt told me in Budapesth that his father had seen Bolyai lop off a spike driven into his doorpost, and that some of his duels were to the death.

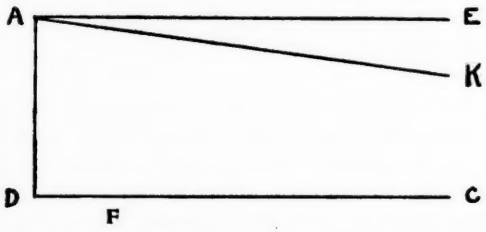
Bolyai published only 24 pages of text, but this is the most extraordinary two dozen pages in the history of human thought. The very first page would kill or cure my genial friend Mr. Russell.

I fear the outcome, since on page 625 of *The Monist,* Vol. XIX, he says: "As to such an idea as that two lines may be parallel if they are taken in the same *sense,* and yet *not* parallel if taken in opposite senses, I fail to find any vestige of it in Lobatchevsky's text. That would be to make Lobatchevsky's system a system of vectors instead of a geometry, and I am sure such a system as well as the idea of a *sensed relation* would put me to permanent intellectual confusion."

The confusion is already here, since vectors assume translation independent of rotation, that is assume the parallel postulate, the whole question at issue.

The Bolyai-Lobatchevsky geometry, then, is of course not a system of vectors, but it is largely a system of *rays.* Bolyai says the ray AB is that half of the straight AB which commences at the point A and contains the point B. Then § 1, "If the ray DC is not cut by the ray AE, situated in the same plane, but is cut by every ray AK comprised in the angle DAE, this is designated by AE||| DC.

It is evident that *there is one such ray* AE, *and only one,* from any point A outside the straight DC, and that the sum of the angles CDA, DAE does not exceed two right angles.



It is also clear that AE|||FC, wherever the point F be taken on the straight DC, supposing in all such cases DC>DF; that is, ray FC of the same sense as ray DC.

Bolyai translates by the word *asymptote* his symbol |||.

To a given ray DC, from a given point A outside it, there is one and only one asymptote AE. Bolyai calls parallel to the straight DC every coplanar straight through A which nowhere cuts the straight DC. So through A, in his terminology, there is only one asymptote to the ray DC, but there may be an infinity of parallels to the straight DC.

Philip Kelland, senior wrangler and tutor to Sylvester, whom

he called the greatest mathematician in the world, was an independent discoverer of the non-Euclidean geometry, and like Bolyai, he called parallel to a given straight all straights coplanar with it which nowhere met it. But this meaning and use of the word parallel has been superseded by Lobatchevsky's. He has no name for this possible infinity of straights through A coplanar with the straight DC but nowhere cutting it. But that one of them which contains the Bolyai asymptote to the ray DC, Lobatchevsky calls parallel to DC and then says "upon the other side of the perpendicular from A will lie also a line AM, parallel to the prolongation DC′ of DC, so that under this assumption we must also make a distinction of *sides in parallelism.*"

This *sensed relation* then is Lobatchevsky's parallelism, and that it is a sensed relation, a one-sided relation, a one-way relation, a relation which goes toward one side only is stressed and emphasized by his § 24: "The farther parallel lines are prolonged *on the side of their parallelism,* the more they approach one another." In other words, Lobatchevsky parallels are one-way asymptotes.

So his next theorem, § 25, which tripped Mr. Russell, I might have translated: Two straight lines which are same-way asymptotes to a third are also asymptotes to one another.

So of all straights coplanar with a given horizontal straight which nowhere cross it, two through each given point are parallel to it, one to the right, the other to the left, and all the others I call *ultra-parallel.*

A parallel to a straight meets it at infinity. An ultra-parallel does not even meet it at infinity. Parallels are straights with a common point at infinity. Parallels are straights which meet on a figurative point. If ultra-parallels determine a point, we must have another name for it. Call it an ideal point. Thus equipped, we are able to answer Mr. Russell's question, *Monist,* page 621: "How is the professional expert (the man who knows non-Euclidean geometry) better fitted to see more lucidly in dealing with the elements of geometry than any other person of good geometric faculty?"

Just thus, my dear friend: You say, "I will now spread before the reader in detail what seems to me to be good geometrical proof of my proposition. Consider and refer to the following figure." I do, and instantly, at a glance, I see your fallacy, your *petitio principii.* I am saved the reading of your three pages of pseudo-proof. You assume that an angle is determined in size because it is made by the side with the hypotenuse of an isosceles right-angled

triangle. You say, p. 624: "Now the angle MEN being a w-angle equals the angle $\beta\epsilon\delta\ldots$"; in other words, you assume that the size of the isosceles right triangle has no effect on the size of its acute angles. This is Wallis's form of the parallel postulate. And so you are guilty of begging the question.

Not only would a smattering of Bolyai have saved you but so would a little excursion into Chapter XV of my *Rational Geometry*, which starts by saying: "Deducing spherics from a set of assumptions which give no parallels, *no similar figures*, we get a two-dimensional non-Euclidean geometry, yet one whose results are also part of three-dimensional Euclidean."

GEORGE BRUCE HALSTED.

GREELEY, COLORADO.

A REMARK ON F. C. RUSSELL'S THEOREM.

F. C. Russell of Chicago has endeavored in the April number of *The Monist* to disprove the legitimacy of the non-Euclidean geometry by showing the demonstrability of the parallel postulate. The basis of his considerations has been laid on a simple proposition that the angle-sum of an isosceles right-angled triangle equals two right angles; a proposition the proof of which he does not dare give, saying it would be "spreading an imputation upon the reader," being so simple in nature. But the whole secret of the matter remained concealed under this unknown sort of a proof, and so we are lucky that we had it imparted to us by Russell himself in a subsequent number. In studying it, we have found all that can be desired.

Russell defines his u-angles "as being such angles as the sides of an isosceles right-angled triangle make with the hypotenuse." This definition is of course not in any way objectionable, but when Russell has to consider the u-angles arising from different triangles of unequal sizes, to be always equal, he has unconsciously fallen into a pit of thought, from which he is unable to get out. When we adhere to the Euclidean world, we can well prove the assumption Russell makes, but how can he protest the legitimacy of it, when he is going to show the Euclidean system to be the sole one that can be relied upon? If he wants to be credited by us, he must first prove the assumption he has made; which most probably he cannot do without having recourse to the parallel postulate or some-

UMI

thing else that may be substituted for it. In a word, Russell has substituted a different axiom in place of the postulate of Euclid. His endeavor and achievement have however left nothing that could make a step towards disarming the pan-geometricians. We stand uninjured on the same ground as before in spite of all the desperate assaults from the strong hand of Russell, who has utterly failed to disground us.

YOSHIO MIKAMI.

OHARA, KAZUSA, JAPAN.

A MATHEMATICAL PARADOX.

The following paradox appears to me to be interesting because it shows how "common sense" breaks down when dealing with a slightly subtle question.

The question to be discussed is: Is the greatest weight that a man can lift the same as the least weight that he cannot lift, or not; and if the weights are different, which is the greater?

The numerical values of all possible weights (both those which the man A can, and those which he cannot, lift at the particular moment considered) form the simply-ordered aggregate of positive real numbers R. Those weights that A can (at this particular time) lift bring about what Dedekind* called a section (*Schnitt*) in R, and all the members of R fall into the two classes:

a. The class of those numbers x such that A can lift the weight x (then also A can lift any of the weights less than x);

b. The class of those numbers y such that A cannot lift the weight y (then also A cannot lift any of the weights greater than y).

Now, as is well known, there is one, and only one, number which "generates" this section, and this number is either the upper limit of the class (a), or the lower limit of the class (b), but not both.

Thus, our answer to the question about the weights is: *Either* there is a greatest weight that a man can lift, *or* there is a least weight that he cannot lift, *but not both.* The paradox lies in the fact that, to unaided common sense, the existence of a limit seems just as, or even more, plausible in both cases or neither as in one

* *Stetigkeit und irrationale Zahlen,* Braunschweig, 1872 and 1892 (English translation in Dedekind's *Essays on the Theory of Number,* Open Court Publishing Co., Chicago, 1901).

only. I cannot see how one is to tell in which case the limit does exist; only that it must in one, and only one, of the two cases.

In my opinion what is paradoxical to the ordinary mind in this is: We have two classes of an infinity of members each (arranged in some order); now ask a person if there is a highest in the first class; if he says "yes" (or "no") he will probably admit by parity of reasoning, that there is (or is not) a lowest in the second.

And yet my case is a translation into picturesque language of an instance well known to modern mathematicians in which the answer *must* be "yes" in the one case and "no" in the other.

PHILIP E. B. JOURDAIN.

BEAMINSTER, DORSET, ENGLAND.

ON THE PROBLEM AND METHOD OF PSYCHOLOGY OF RELIGION.[1]

In a report before the Congress of Psychology at Geneva Prof. Harald Höffding of the University of Copenhagen undertook to sum up his theory of the psychology of religion with, we must admit, an air of easy and careless assurance. In such a delicate investigation we can not say, "I am right"; much less, "You are wrong." I am not writing at all in this spirit, and I recognize in Professor Höffding too great a degree of culture to assume it in him. However, a fear has taken possession of me and I have not succeeded in freeing myself from it. This is the fear lest Professor Höffding does not take into account so much as they deserve certain difficulties which consciousness raises against the dogmatic presuppositions which form its point of departure,—difficulties which I do not pretend have been solved and much less do I pretend to solve them myself, but whose proper comprehension will always be one step forward.

Professor Höffding's entire conception rests upon the postulate which he lays down as most natural, that the psychology of religion is a part of general psychology. However, religion does not lend itself readily to this classification, but if it did it would be so much like other questions, that if a psychology of religion existed, its first claim would be that general psychology forms a part of the psychology of religion. For it is entirely gratuitous and arbitrary to consider "religion as a particular form and a particular direction

[1] Translated from the French of Professor Billia by Lydia G. Robinson.

of psychological life." But what constitutes religion in consciousness is precisely the negation of a particular form and a particular direction. Do you wish to eradicate something from your cognizance of a thing? Explain it by another while forgetting the proper and constitutive element it possesses. Had I not a horror of all psychological Baedekers I would say, "Beware of false generalizations." Yes, I learned at an early date, and I still maintain to-day, that nothing is apprehended except in that which is more general and more common. But there is the general and the generic. That with which we have to deal is the general that is called the universal, that does not exclude the essential, the particular, but includes them. This difficulty does not entirely escape Mr. Höffding who endeavors to save himself by analogy. The objection suggests itself that religion must be cognized by its own experience in order to apply psychology to it. He expects this objection and thinks he can overcome it by saying that in default of this experience the psychologist would be able to conceive religious phenomena while seeking to discover what place religion would occupy in his psychical life if it existed for him, in the same way that he might be able to imagine colors beyond red or violet. For this it would be necessary for him to direct his attention to the place which religion occupies in the psychical life of those who are acquainted with it from their own experience. The question is to define the psychological place of religion in the same way that the geometrical locus of a point is defined in mathematics. But there is no psychological locus and least of all for religion. Religion does not occupy a place in the psychical life but it is the whole. It includes and creates all. When it is anything else, it is either no longer religion or it is not yet religion. Religion is quite a different thing to those who are acquainted with it than to those who do not know it. For instance certain rites, sacraments, prayers, ecstacies, clash too much with so-called reason to be comprehended by an indifferent person as they actually exist in a believer who experiences that actual perception upon which alone religion rests. It is like love. Nothing is more absurd than to see an indifferent person judge or laugh at it; as if the external manifestations were the same thing for the one who experiences the passion as for others, and especially as if the principle from which these manifestations arise is in any way comprehensible except to him who has the experience. It is just such a monstrous absurdity as the attempt to impose upon universal culture a history of philosophy conceived and drawn up apart from all

philosophy. It is because of this, that in the analogy with which Mr. Höffding appears to be content, what is still to be accomplished is given as accomplished. It is because of this that, contrary to the opinion of Mr. Höffding, I think that the historical method has no value for the psychology of religion; that it is very useful as a curiosity but if it takes a place ahead of psychology it only leads it astray and gives it the illusion of describing and cognizing a mental fact while remaining outside of the fact itself, which, especially for a psychologist is too great an evil. In fact it is here indeed that Mr. Höffding takes refuge. The psychology of the savage and the primitive man to which he resorts, following the example of Westermarck, is so comfortable! "Magic," he says, "is a more primitive phenomenon than religion." How does he know? "Religion," he adds, "does not appear until man finds out by experience the insufficiency of his magic power; that is to say, until he perceives his own limitations and his dependence. Hence it is by the path of resignation that the step is taken from magic to religion." Although he raises before our eyes a very imposing "hence," I have heard a gamin murmur, "And who told you so?" I do not deny the rôle which imagination plays. It is very useful in history and becomes an absolute necessity in pre-history. But in this case instead of imaginary experiences we have experiences which are truly experimental; actual experiences established which tell us that it is not by resignation that we reach religion, but it is by religion that we pass from a vain effort and despair to resignation.

One word more. What Professor Höffding has to say about the critical period at which the religious element becomes detached from other forms of the mind is very instructive, but I may be permitted to add that this does not detract from religion but advances it. When the so-called other sides of the spiritual life become detached from religion it is well for religion to be separated from them in order to accentuate its superiority, its spirituality, and essential inwardness.

The fundamental objection which I brought against Professor Höffding is so simple that it is not surprising that in the Congress of Philosophy it should have been heard from several quarters. The result was that by the second session I said no more, because the idea for which I stood had found in the meantime far more able interpreters than I in other speakers, especially M. Lutoslawski. But since I did not stay to the end of the discussion which extended through three or four other extra sessions, I was not aware that

UMI

any one else had undertaken to give a trenchant answer to that objection which in my simplicity I considered unanswerable. I have read of it in the report of the Congress published in the *Revue de philosophie* of September, page 430. In replying to M. Rochat, who, almost like myself, refuses to admit that a psychologist deprived of religious sense[2] can study religious phenomena to advantage, M. Bernard Leroy, of l'Ecole des Hautes Etudes, declared that one need not be religious to speak scientifically about religion, any more than he need be mad in order to discuss mental pathology. Profound observation! Astonishing discovery! It deserves my fullest appreciation. Oh, if I had but heard it some years ago! I was once speaking with the greatest unconcern about the self-styled psychiatrists, and with my usual abruptness classed them all in the category of charlatans, and scouted without mercy the pretence of judging the minds and souls of other people, of perceiving from without and of arbitrary classifying mental disease by means of the hastiest inductions, when a young physician of my audience regaled me with the classical classification accepted by the modern schools of sufferers from mental diseases into imbeciles, psycho-neurotics, degenerates, psychotoxics and encephalopathics, with their subdivisions into melancholics, maniacs, etc., etc., in such convincing accents that I was reduced to silence, and came to the conclusion that a man not versed in psychiatry should not attempt to speak of it so lightly. Simpleton that I was! If I had then heard M. Leroy I would have known how to retaliate, and could have answered that it is no more necessary to be a psychiatrist in order to talk about psychiatry, than it is to be a charlatan in order to talk about charlatanry.

Yet the opinions of Professor Höffding and of Mr. Leroy are instructive from another side. They show us the range of a very common view of which we are not always fully aware and which I hope to surpass. The postulate that the psychology of religion is a part of general psychology depends upon the postulate that psychological observation is a part of general observation, which may be divided into the two classes, psychological or internal, and physical or external. One rather cautious party of spiritualists thinks that it has already attained and safeguarded a good strategic position by affirming the distinction between the two kinds of observation

[2] I say "*almost* like myself." As far as I am concerned, what I have said above leaves me no room to admit that a *psychologist* is ever deprived of a religious sense.

and the rights of the psychological in relation to the physical, which, although more precise and more definite, is not the only kind. But this is not enough for me.

What is physical observation if not the observation of luminous, audible, extended,—that is to say, sense-perceptible, phenomena; and what are sense-perceptible phenomena if not perceptions, that is to say, one kind of our psychical states? Physical phenomena, including phenomena of extension, are clearly distinct from psychical phenomena. Yes, but there are not two different observations but only one; psychology is not a kind of observation but observation itself. But I have just shown that psychology is not curiosity but improvement, that one can not observe a fact without producing it,[3] and that psychology is more derived from ethics than ethics from psychology. By giving ourselves this point of view we shall no longer separate the religious character of observation from religiousness. As soon as improvement takes place through internal observation, and the observation is made from a sense of duty, ethics and religion, the loftiest possessions of the soul, will no longer be kinds of activity or kinds of observation, but the very principle of activity and of observation.

L. MICHELANGELO BILLIA.

TURIN, ITALY.

THE PRAGMATIST VIEW OF TRUTH.

A PROBLEM WITHOUT A SOLUTION.[1]

Pragmatism is still agitating the philosophical world, and Prof. William James continues the good fight dealing blows right and left. There is a change only in so far as pragmatism does not seem to spread further, and its ingenious leader now assumes more and more the defensive. His main weapon consists in the declaration that his antagonists have misunderstood him. They are accused of distorting his views into silly absurdities which he did not mean to say, and they are put down with such phrases as, "this is the usual slander" (p. 274).

In our criticisms we have always been careful to quote the

[3] "Has the Psychological Laboratory Proved Helpful?" *Monist*, July, 1909. Compare "La Philosophie c'est l'unité morale, *Bericht d. III. Internat. Congr. für Philosophie, Heidelberg.*

[1] A review of *The Meaning of Truth, A Sequel to Pragmatism*, by William James, New York: Longmans Green & Co. Price $1.25 net.

master's *ipsissima verba,* and so we feel that his complaint is not applicable in our case.

But pragmatism is so subtile that no one appears to be able to appreciate it unless he enters into its spirit with enthusiasm. Professor James says (pp. 183-184):

"The pragmatist question is not only so subtile as to have escaped attention hitherto, but even so subtile, it would seem, that when openly broached now, dogmatists and sceptics alike fail to apprehend it, and deem the pragmatist to be treating of something wholly different."

The difficulty of understanding pragmatism is greatly increased to outsiders, to intellectualists as they are called, to rationalists, to monists, and to the whole crowd of anti-pragmatists, by the brilliant dicta of Professor James, who in his zeal sometimes makes statements which he does not mean and which he offers only as an olive branch to please antagonists or to gain their good will. Professor James says in the preface:

"One of the accusations which I oftenest have had to meet is that of making the truth of our religious beliefs consist in their 'feeling good' to us, and in nothing else. I regret to have given some excuse for this charge, by the unguarded language in which, in the book *Pragmatism,* I spoke of the truth of the belief of certain philosophers in the absolute. Explaining why I do not believe in the absolute myself (page 78), yet finding that it may secure 'moral holidays' to those who need them, and is true in so far forth (if to gain moral holidays be a good), I offered this as a conciliatory olive-branch to my enemies. But they, as is only too common with such offerings, trampled the gift under foot and turned and rent the giver. I had counted too much on their good will—oh for the rarity of Christian charity under the sun! Oh for the rarity of ordinary secular intelligence also!"

Professor James complains about "the rarity of Christian charity" and "the rarity of ordinary secular intelligence." But is he not guilty of the same fault when he misconstrues what other thinkers have said before him; when he censures them in sweeping assertions for mistakes of which only some of them are guilty; when for instance he declares (p. 192) that "throughout the history of philosophy the subject and its object have been treated as absolutely discontinuous entities" (p. 136); while we know that almost every philosopher has considered the two as correlates? If our pragmatists were more familiar with the history of philosophy they would probably not boast so loudly of the originality of the movement, the leading ideas of which are old errors.

We do not doubt that Professor James has been frequently misunderstood, and he confesses himself that he did not always

mean what he said, but it appears that the main reason that he is so much misunderstood is his own carelessness. On page 272 Professor James says with reference to the criticism of Professor Bertrand Russell:

"When, for instance, we say that a true proposition is one the consequences of believing which are good, he assumes us to mean that any one who believes a proposition to be true must first have made out clearly that its consequences *are* good, and that his belief must primarily be in that fact, —an obvious absurdity, for that fact is the deliverance of a new proposition, quite different from the first one and is, moreover, a fact usually very hard to verify, it being 'far easier,' as Mr. Russell justly says, 'to settle the plain question of fact: "Have popes always been infallible?" than to settle the question whether the effects of thinking them infallible are on the whole good.' We affirm nothing as silly as Mr. Russell supposes."

We are glad to know that Professor James does not mean to make the pragmatic result of a belief the test of its truth; but we can not help thinking that his explanations of the meaning of pragmatism go pretty far to justify Professor Russell in thinking so. When we refuse to accept pragmatism we also may be under misapprehension; but if words mean what they say, Professor James believes that science is not possible, or at least that what is called science is not reliable, that new fangled theories have replaced the old orthodox conceptions, that Euclid is antiquated because Bolyai and Lobatchevsky have excogitated other geometrical systems, and that truth and its exponent science have neither stability nor objective significance. We may misunderstand Professor James, but this is what he says on page 57:

"As I understand the pragmatist way of seeing things, it owes its being to the break-down which the last fifty years have brought about in the older notions of scientific truth. 'God geometrizes,' it used to be said; and it was believed that Euclid's elements literally reproduced his geometrizing. There is an eternal and unchangeable 'reason'; and its voice was supposed to reverberate in *Barbara* and *Celarent*. So also of the 'laws of nature,' physical and chemical, so of natural history classifications—all were supposed to be exact and exclusive duplicates of pre-human archetypes buried in the structure of things, to which the spark of divinity hidden in our intellect enables us to penetrate. The anatomy of the world is logical, and its logic is that of a university professor, it was thought. Up to about 1850 almost every one believed that sciences expressed truths that were exact copies of a definite code of non-human realities."

Now we deny that geometricians ever believed that Euclid's Elements "literally reproduced God's geometrizing"; or, what means the same, that geometry is a direct description of objective space-

UMI

conditions. All mathematical propositions are purely mental constructions by the aid of which we can calculate the relations that obtain in space, or other conditions, proportions, probabilities, etc., and *mutatis mutandis* the same is true of logical syllogisms and of the laws of nature. None of them are copies or duplicates, or archetypes, but they are formulas by which we comprehend reality and which serve us to adjust our conduct. Here Professor James is guilty of an obvious misunderstanding of the import of science, and he misconstrues the meaning of former thinkers.

While to some extent the pragmatist fights windmills which he takes for giant errors, he takes new fads seriously or exaggerates the importance of new theories, making out that they upset and antiquate all previous science. Professor James continues:[2]

> **"The enormously rapid multiplication of theories in these latter days has well-nigh upset the notion of any one of them being a more literally objective kind of thing than another. There are so many geometries, so many logics, so many physical and chemical hypotheses, so many classifications, each one of them good for so much and yet not good for everything, that the notion that even the truest formula may be a human device and not a literal transcript has dawned upon us."**

The subjectivity of geometry is also insisted upon on pp. 83 ff. On page 85 we read a sentence which reminds us of Kant. Here Professor James says: "The whole fabric of the *a priori* sciences can thus be treated as a man-made product"; though Kant would say that space is "ideal," which means belonging to the domain of ideas, and we would prefer to say, that the *a priori* is "mental or a mind-made product." How ideal or purely mental constructions can possess objective values I have set forth in my book on *Kant's Prolegomena.*

But in the pragmatist conception everything dwindles down to "purely human habits" (p. 29).

A genuine scientific truth is a formula which describes the essential features of a group of facts. A scientific theory is a tentative explanation of facts. Everybody knows that theories and hypotheses are preliminary and we must always be prepared to surrender them. No scientist will regard the change of a theory as a "breakdown" of the notions of scientific truth, be they old or new, but while theories change, truths remain forever. Those features of facts which remain, the "uniformities of nature" as Clifford called them, those eternalities of existence which make science possible, are not subject to change. They are the *raison d'être* on the

one hand of the cosmic order, and on the other hand of man's rationality.

Professor James calls his new book "The Meaning of Truth," but the reader, with the exception of his most ardent admirers, will not know more about what truth in pragmatism means after having read these latest explanations. Professor James even admits that the very "name 'pragmatism' with its suggestions of action, has been an unfortunate choice" (p. 184), and we at any rate must confess that we are more bewildered than helped. Professor James himself says (p. 215):

"As I look back over what I have written, much of it gives me a queer impression, as if the obvious were set forth so condescendingly that readers might well laugh at my pomposity. It may be, however, that concreteness as radical as ours is not so obvious. The whole originality of pragmatism, the whole point in it, is its use of the concrete way of seeing. It begins with concreteness, and returns and ends with it."

Other philosophers too have proposed to begin with concreteness and to end with concreteness, but the worth of a philosophy consists in the method of dealing with the concreteness of existence; yet this portion is missing in pragmatism.

Professor James overestimates the significance of sentiment and underrates the importance of the intellect. His world-conception might be characterized as a philosophy of mood, of temper, of feeling, of subjectivity, in rebellion against the rigid demands of intellect, of science, of consistency of system. He dislikes theory and system, prefers pluralism to monism, clings to the concrete, and abhors the abstract. Such is the poet's and artist's temperament, which is desirable in literature, in lyric effusions, in paintings, especially in *Stimmungsbildern,* but out of place in science and in a scientific philosophy. Such temperamental expressions are perfectly legitimate, and we enjoy the writings of Professor James as such, but we must demur when he parades his subjectivism as philosophy, indeed as the one philosophy to the exclusion of an objective or a scientific philosophy.

Here is a sample of pragmatic epistemology:

"A feeling feels as a gun shoots. If there be nothing to be felt or hit, they discharge themselves *ins Blaue hinein.* If, however, something starts up opposite them, they no longer simply shoot or feel, they hit and know.

"But with this arises a worse objection than any yet made. We the critics look on and see a real q [quality] and a feeling of q; and because the two resemble each other, we say the one knows the other. But what right have we to say this until we know that the feeling of q means to stand for or represent

just that *same other q*? Suppose, instead of one *q*, a number of real *q*'s in the field. If the gun shoots and hits, we can easily see which one of them it hits. But how can we distinguish which one the feeling knows? It knows the one it stands for. But which one *does* it stand for? It declares no intention in this respect. It merely resembles; it resembles all indifferently; and resembling, *per se*, is not necessarily representing or standing-for at all. Eggs resemble each other, but do not on that account represent, stand for, or know each other. And if you say this is because neither of them is a *feeling*, then imagine the world to consist of nothing but toothaches, which *are* feelings, feelings resembling each other exactly,—would they know each other the better for all that?"

Rambling but witty, full of misconceptions but entertaining, and disposing of the problem with a joke,—such is the style of the leader of the pragmatic movement.

The book talks about truth, but never and nowhere does it clinch the problem. We grant that it combats many errors, although we must add that frequently what it combats are but straw men of the author's own making. But whatever errors pragmatism may be guilty of, Professor James is man of great vigor and ingenuity. Though we would say that Professor James makes serious blunders and is sometimes unfair to his antagonists, though he misconstrues the philosophies of the past, though he lacks clearness of thought, the first requisite for a philosopher, his writings possess a charm that is unrivaled. He may be wrong in all his contentions, but he is never dull.

EDITOR.

WILLIAM PORCHER DU BOSE, A CHRISTIAN PHILOSOPHER.

Philosophy and religion cover practically the same ground; both are world-conceptions which serve for the guidance of conduct. Both undertake the great task of explaining the nature of existence, of pointing out its significance and of deducing the lesson which it teaches. Both appeal alike to the head, the heart, and the hands. They appeal (1) to the intellect in offering a definite doctrine as to the facts in question; (2) to the emotions in determining the desired attitude toward existence as a whole; and (3) to our will in rousing our energies and impelling them to the approved kind of action.

The difference between philosophy and religion is this, that philosophy is the world-conception of a thinker, and religion the

world-conception of an historic movement. As such it may be characterized as the philosophy of a community, viz., of a congregation of people, not of one individual but of a multitude, of a state, of a nation, a group of nations, of a Church or whatever kind of community this superindividual movement may embrace. We purposely use in our definition the term "historic movement" instead of "community" or "congregation," because a religion is not limited to one set of contemporary persons, but extends over many generations. Religious beliefs (and this is practically true of all ideas) possess a superpersonal life. They may be born in the soul of one man, but they are transferred to others, and pass down from father to son, from teacher to disciple. In their passage from one generation to another, and in their transference from one nationality to another, they change, they are modified, they are seen in another light, they receive new interpretations and new applications, they grow and develop, sometimes they wither, but the modifications which they undergo are continuous. In a word, each one of them represents a movement in the history of mankind; and all of them (i. e., all the beliefs and ideas of a special religion) constitute a great superpersonal organism, the development of which is subject to law and presents many analogies to the growth of physical organisms.

During the Middle Ages Christianity had become the world-conception of the nations of Europe and so almost all philosophers of that period are expounders of Christianity. This is especially true of Thomas Aquinas, who was a man of unusual breadth for his age, assimilating the non-Christian learning accessible to him, viz., the Aristotelian philosophy, to Christian doctrine and incorporating it into the system of the Christian world-conception. Since then Christianity has grown and entered into a new phase during the sixteenth century. In the meantime philosophy as a secular world-conception began to develop gradually in independence of the Christian religion. To-day Christian philosophers are an exception.

By "Christian philosopher" I understand a thinker who like Thomas Aquinas accepts the Christian world-conception as handed down by tradition, uses its doctrines and symbols, and adapts it to the general knowledge of the age. Such a Christian philosopher of the Roman Catholic Church is Rosmini and of the American Episcopalians, William Porcher Du Bose.

Professor Du Bose is now in his seventy-fourth year and lives in the vicinity of the University of the South, the center of learning

UMI

of the Protestant Episcopal Church in America. He was born in Winnsboro, S. C., in 1836. He was graduated as Doctor of Systematic Theology from the Univesity of Virginia in 1859, and served as adjutant and later as chaplain in the Confederate Army during the war. He was severely wounded and taken prisoner. He was supposed to be dead, but recovered, and was spared to devote his energies on the return of peace to the interests of education. He entered the ministry in 1865, being rector successively in Winnsboro and Abbeville, South Carolina. He became in turn chaplain, professor, and finally dean of the theological department of the University of the South.

Professor Du Bose has not only found recognition in his own country but also abroad. Professor Sanday,[1] of Oxford, calls him the "prophet of Sewanee" and "a real sage and seer" (p. 644), and in another place "a religious philosopher." He characterizes his book, *The Gospel in the Gospels*, as "rather a sustained endeavor to get at the inner spiritual meaning that lies behind all external presentation" (p. 644), and he takes note of the fact (p. 645) that "Du Bose is very sympathetic toward modern thought" and that "in view of the present condition a different attitude is advisable from that which was characteristic of early Christianity."

The large mass of philosophers and people interested in philosophy have become estranged from both religion and the Church. They can scarcely understand the attitude of a man whose philosophy is decidedly Christian. They are apt either to think that the thoughts of a Christian philosopher are absolutely unscientific and even anti-scientific, or suspect that his Christianity is no longer genuine. The truth lies midway. Men like Rosmini and Du Bose are quite scientific and certainly not anti-scientific, and yet their Christianity has remained as genuine, at least, as the Christianity of Thomas Aquinas after that was broadened by contact with the culture of classical antiquity. Whether or not these thinkers will become prominent factors in the further development of mankind is another question which we shall not venture to enter upon, though we may assume that most of our readers, even confessed Christians, will answer in the negative. We will here limit ourselves to an exposition of the philosophy of Professor Du Bose, and we propose to give it as much as possible in his own words.

Four volumes lie before us (all published by Longmans, Green & Co.), entitled *The Soteriology of the New Testament* (1906).

[1] In *The Churchman* for April 28, 1906.

The Gospel According to St. Paul (1907), *The Gospel in the Gospels* (1908), and *High Priesthood and Sacrifice* (1909), and all of them are written in the style and world-conception of the Christian faith. We do not claim that the meaning of traditional terms is the same as that of the Apostles, or of St. Athanasius, Thomas Aquinas, or Rosmini, but we do say that it is a faithful continuation of the stream of thought which in its entirety is called the Christian religion. Du Bose is a thinker of catholic tendencies, by which expression we mean that he endeavors to remain within the movement which has produced him and from which he draws all his inspirations. There are other thinkers who would emphasize differences for the sake of breaking away from their own tradition and branching off into a new direction often with an ostensible show of originality. Not so Du Bose, and yet his interpretation of Christian doctrine is original enough, and his views bear the imprint of a modern education far more than may at first sight appear. He says:

"For each time must have its own living interpretation, since the interpretation cannot but be, in half measure at least, relative to the time. If the divine part in it is fixed, the human is progressive and changing just in so far as it is living."

He has studied the past and has sifted in it that which is transient from that which is eternal. He describes his task thus:

"We are attempting to address the world to-day, in the matter of its profoundest interest, in terms of the world two thousand years ago. We have first to know what those terms meant then, and to prove that all they meant then they mean now, and mean for all men in all time."

This method "brings with it its embarrassments." Though the times have changed, the Bible and the creeds are not antiquated, and

"We must take measures to preserve them, and the only way to preserve them is to make them as living to day, as much part of our thought and our speech and our life now, as they were two thousand years ago.

"In order to do that, we must cease to treat the phraseology, the forms, definitions, and dogmas of Christianity as sacred relics, too sacred to be handled. We must take them out of their napkins, strip them of their cerements, and turn them into current coin. We must let them do business in the life that is living now, and take part in the thought and feeling and activity of the men of the world of to-day."

Professor Du Bose is apparently well acquainted with the results of church history and of higher criticism, but he makes little use of them. He seems to know them, but they are trifles. He cares for the world-conception that lies behind the Scriptures and of

which they have become different expressions. He attributes to Paul the great work of having preached the pre-existent Christ and of thus having laid the cornerstone of the system of Christian doctrines. What harm that the Gospel writers have not yet realized this ideal of the God-man, if it can be seen to be founded in the eternal nature of things? Du Bose says, "The Scriptures are not Christianity, but themselves a product of Christianity"—*Soteriology*, p. 20. And whatever discrepancies there may be in the Gospels, they are after all different aspects only of one and the same gospel, and this gospel, viz., the gospel in the Gospels is the precious ideal we should care for; it is the same of which St. Paul rendered later a more complete construction.

Du Bose accepts the doctrine that mankind and every single man is the son of God. He says:

"It is an old and familiar issue among us: Did Jesus Christ find man son of God, or did He make him so?....I shall not so much undertake to decide between these two views as attempt to state the truth of both."

It will be interesting to know what the person of Christ is to a man like Du Bose, and we formulate the question thus: Why is Jesus to him the Christ, the Saviour, the Incarnation of God? His answer is this:

"Jesus Himself in His humanity needed the salvation which all humanity needs. Salvation for Him, as for us demanded that conflict with sin and conquest of sin which was preeminently His experience and His achievement. Salvation for Him as for us was impossible either as a mere fact of nature or as an attainment of His own human will. The power of God unto salvation through faith was necessary for Him as for us, and that power manifested itself in Him, as it must in us, in the perfection of His human obedience unto death, even the death of the cross. Through His own blood He entered into the holiest, having found eternal redemption. That is to say, through the perfect loss or offering up of Himself He eternally found or attained His true Self."

"The joy of truth cannot lie in its individual possession. One cannot think of the enthusiasm of Newton's discovery as a possession and pleasure confined to himself. What was it for him, if it was not for the world? Truth, beauty, goodness are infinite personal possessions too, but they are all so in the degree in which they are shared, and are impossible or valueless as only one's own....The holiness, the righteousness, the eternal life which our Lord had achieved was His own; but, O, to Him how little was it, how much was it not, His own! To have been the Truth, and have the truth that He was denied; to have created righteousness, and have the righteousness rejected; to have been all Love and Goodness realized and manifested, and to be met with hate and requited with evil! The contradiction, the gainsaying of sinners against themselves, against their own souls, was what Jesus had to endure,—why? Because He was their true Self,—all the reason and the meaning and

the justification, all the truth and the beauty and the goodness or good of themselves. Himself alone was not the true self of Jesus; He included all selves in Himself, and suffers, and is crucified and put to shame, or lives anew, rejoices, and is glorified in the whole body and in every member of the humanity that is Himself and His own."

Of salvation Du Bose says that "the term itself gives no hint what it is." He says:

"The word Salvation comes over into the New Testament from the Old, and it comes freighted with many associations which invest it with something of a figurative and poetical character. As, e. g., in 1 Cor. xv, where death is personified as the last of a number of enemies to be overcome and destroyed in the Messianic Salvation. The more external and temporal Salvations of the Old Testament, which familiarized the mind to the great truth of God as Redeemer and Saviour of His people, are generally from personal enemies, as Pharaoh, the heathen, and especially the great world powers which successively oppressed them. The idea of Salvation had finally shaped itself into the definite hope of the Messianic redemption, the expectation of a divine deliverer who should save them from all their enemies. But who and what were these enemies? We may be sure that when we have passed beyond the mere outward figures and symbols of things to the things themselves, when we come to do with that which is Salvation indeed, it shall be from those things which are enemies indeed. And has not humanity its enemies? And what are these in the last analysis but ignorance, and sin, and death! No merely outward enemies are in themselves enemies. Pharaoh was the helper of Israel's redemption as Judas was of our Lord's exaltation. The devil himself is to God's saints a ministering spirit sent forth to minister to the heirs of Salvation. Naught in this universe hurts or can hurt us but what we ourselves are and do. All things work together for our good, save as we ourselves defeat them and are our own enemies. What we need to be saved from is ourselves; and our only Salvation is that death of ourselves which is the life of ourselves. The cross is the only instrument of human Salvation."

Our own incompletion is death and Christ is life. Says Du Bose:

"Now 'as in Adam all die even so in Christ shall all be made alive.' No one can know his New Testament without knowing how universally and consistently our relation to Christ is represented as a being *in Christ*. The Incarnation is not an individual fact limited to our Lord as a man; it is a generic fact including, or intended to include, in Him the whole race of man. I hope to prove and illustrate at length elsewhere the important truth that *all* that He was and did in the body of His particular incarnation He is and does, or is to be and do, in the greater body of His general or generic incarnation; that as He has glorified humanity in His own individual body, so is He to glorify it in the great body of his saints, who are only such as they are in Him and He in them. This mode of representation so pervades the New Testament that we need only to allude to the numerous descriptions of the Church as the Body of Christ—described also as His πλήρωμα, or fulness, 'that which is filled with His presence, power, agency, and riches.'"

Typical of Du Bose's mode of thinking is his explanation of the significance of resurrection. He has nothing to say about the physical event, "the resuscitation of a dead body." True he says nothing against it, but he leaves it alone. His interest is concentrated on its significance.

He says:

"The resurrection, then, means to us so infinitely more than the physical or physiological puzzle of the resuscitation of a dead person, that the acknowledged and perhaps insoluble difficulties involved in that practically do not disturb one who appreciates and measures the spiritual significance and necessity of the fact. Christianity has permitted itself to be so mixed up with and embarrassed by the natural aspects of the case, that it has weakened its grasp upon the true fulness and incontestability of the spiritual truth and proof of the resurrection. For my part, and I think in the interest of spiritual rather than of physical science—though I believe them to be one—I fully share the current prejudice against mere miracle (at any rate as we have been understanding it) as explanation for any phenomenon. I should very much prefer to believe that in what we call the miracles of our Lord, and especially in the momentous fact of the resurrection, there is manifested some higher natural working than we have as yet been able to correlate with what we so far know of nature. I am loath to believe that what I consider the most significant, beneficent, and interpretative event in creation should have been interjected into it as an interference or amendment. But at any rate Christianity, I think, can afford to leave to a lower science what of puzzle there is in reconciling the differing and often seemingly conflicting spheres of the spiritual and the physical in human experience. The problem, for example, of the reconciliation of personal freedom and natural causation will probably never be solved, and yet the facts will forever continue.

"What then, let us recall, is the fuller significance of the resurrection? As the death of Jesus, in its spiritual aspect, was not the fact of a moment but the act of a lifetime, as the cross went with Him from the cradle to the grave, and through every minute of every day—as also, He said, it should accompany us,—so also was the resurrection of our Lord a continuous and unbroken act and fact of His whole life. It was a consistent breaking through or transcending the limitations that bind "all us the rest" in the universal subjection to sin and death. The work of Jesus was the fact of His holiness, and every moment of His holiness was an act of resurrection, inasmuch as it was a raising our common nature out of and above its natural state or activity. The death habitually spoken of in the New Testament, at any rate in its higher teachings, is not a physical event. It may and does involve that too, sooner or later, but even physical death, strictly as such, always presupposes an interior spiritual death. Not, I think, that even St. Paul believes that but for the entrance of sin there would not have been the natural change of death; only that that natural change would not without sin have been the dark thing we now, in consequence of sin, know as death. Rather would it have been a change and an awakening, a second birth into a higher life. Sin is not the cause of death as a natural change, it only makes it death in the unnatural evil and dread of it. So it is only the sting and curse of death. Extract the

sting, remove the curse, and death ceases to be death in its bad sense, and becomes only a release and rest from the sorrows of this world and a blessed entrance upon the activities and joys of another. And that other is not a future world only, but an ever present one. It is the kingdom of God or kingdom of heaven which was established in this world by our Lord's life work in it. It is the kingdom of which He Himself said that no one could see it or enter into it except by a new birth from above, a birth which is potentially the whole of the death to sin and the resurrection to holiness and God."

Du Bose makes a distinction between the terms "real" and "literal," and applies them to his interpretation of the Lord's Supper. He says:

"In speaking of what I have called the literal and real truth of the language of the sacrament, I have, perhaps, failed to make a distinction which ought to be made between the terms *literal* and *real*. 'This is my body,' may express a very real without expressing a literal fact. The doctrine of transubstantiation seems to me to assume that the *real* truth of our Lord's words is inseparable from their *literal* truth. The doctrine of real presence assumes a reality which is separable from mere literalness, and which is not less real and far higher for being so separated. Every *verbum Dei* must *be* what it *means*, but it must be so in the sphere, or order of being, in which it is meant or intended....Our Lord Himself spoke of our continuing to see Him, hear Him, etc.; and we do not doubt that we do so in a very real, although not in a literal, sense."

Further down Du Bose explains his interpretation of the sacrament:

"But still the question remains, What are 'the body and blood of Christ,' which are objectively or divinely given, and subjectively or humanly received, in the Lord's Supper? What is it which under the terms 'body and blood of Christ,' is verily and indeed given and received?....

"If this question is to be answered in the light of the whole New Testament teaching, I think we must say, first, that what is given and received in the Lord's Supper is *Jesus Christ Himself*, as not only our new lives, but our new selves, as the ideal and spiritual self which in every man is to take the place of his actual and carnal self. As once for all, in baptism, we put off ourselves and put on Christ, so, in every communion, we repeat that act. For while in the grace of God, and in our own answering faith, we are once for all passed out of ourselves into Christ, yet in fact we need constantly by the grace of God to be putting off ourselves and putting on Christ. Baptism may be said to correspond to what is now called our justification, and the Holy Communion to our sanctification. The one is the once-for-all identification of us with Christ, and the other is the gradual and progressive identification of us with Christ."

We receive Christ and Christ means the higher life, the life of sacrifice. Du Bose says:

"He imparts to us the powers of the *new humanity* which He has insti-

tuted, by becoming in us the holiness, the righteousness, and the life of God. Secondly, the *flesh and blood* given us in the Holy Communion are flesh broken and blood shed; it is not only humanity, but humanity in sacrifice, brought into covenant with God by sacrifice."

We have in Du Bose's exposition a Christianity which leaves out the literal aspect of the dogma. He neither rejects nor refutes it, nor does he accept it; he simply ignores it and lays all stress upon the significance of Christian ideas. It can not be said that he hollows them out and leaves us the empty shell, for on the contrary, he does not throw away the shell, but opens it and shows us the kernel.

We do not venture to decide what part a Christian philosophy will play in the future. Christians may discover that the same truths which underlie Christianity can be obtained by other means and that ultimately they agree more or less with the wisdom of the sages of all climes and nations. There is an ideal toward which all converge, Zarathustra, Lao-tze, Buddha, and Confucius, and even Mohammed. On the other hand we trust that the critics and enemies of Christianity will gradually learn to judge it not from its shell but its kernel.

Friends of Professor Du Bose at the University of the South in Sewanee, Tennessee, have published a calendar in his honor which contains twelve characteristic passages from his books. They read as follows:

"The Cross is the eternal symbol of selfsacrificing love, and love in which God lost and found Himself in us and in which we lose and find ourselves in God.

"Christianity gives us all things, but it requires of us absolutely all the things which it gives us.

"Nothing can dispense us from the humble and devout use of divine means, except the fact of having through their appointed use as means attained the ends for which they were instituted.

"When you have made the law as high as God Himself, you will want God Himself in you to enable you to fulfil it.

"It is not the receiving, but the showing mercy, not our being forgiven, but our forgiving, that Jesus Christ is concerned about.

"Knowing Jesus Christ is our gospel and being what He is is our salvation.

"We may give ourselves in many ways and in many degrees, but it is never real sacrifice unless its spirit is love and its form is mercy.

"Repentance, if it is repentance at all, must repent of sin as sin and of any and all sin.

"As God sees us, as He has eternally foreseen and purposed us, so has He manifested us to ourselves in Jesus Christ.

"The Son of Man took to Himself no special privilege of birth or wealth or class or office; He stood upon His manhood.

"The pure in heart see men as well as God as they are, and have the sweet reasonableness to deal with them as they should.

"It is not the being loved but the loving with a divine love that is our salvation."

Though Professor Du Bose could not be called a mystic, a careful reader will find many points of resemblance between his mode of approaching theological problems and that of such men as the author of *Theologica Germanica,* Tauler, and Angelus Silesius.

EDITOR.

PSYCHOLOGY MORE THAN A SCIENCE.[1]

Psychology, or the study of the soul, can have its origin only within that very consciousness which is at the same time the object of its investigation. It is for this reason that psychology has been considered a branch of philosophy and has shared the same lot, now honored and now under suspicion, banished. It is a well-known fact that psychology has been banished by rigorous positivism. Perhaps in so doing extreme positivism has been logical without being aware of it.

If by science is understood the classification of facts, and if by facts we mean those perceptible to the external senses only, psychology has no place in science or among facts. A superficial observer would be tempted to attribute the abandonment of psychology to the triumphant taste for natural science. Nothing is farther from the truth than this. It is either a slander or an over-valuation, whichever you wish to regard it.

If materialism is to be approved because it is a system, we could not legitimately derive a system from the taste for and the habit of studying natural facts that are scattered, divergent, and without unity. The process of reducing to a unity, even to what spiritualists and critics consider a false unity, belongs always to the province of metaphysics; it always exceeds the bounds of facts. If on the other hand you say that the exclusion of psychology is to be condemned, nothing is more incorrect than to attribute it to natural science which might easily be accused of confusing psychical with

[1] Address delivered at the sixth International Congress of Psychology, held at Geneva, Switzerland, in August, 1909, and translated from the French of Professor Billia by Lydia G. Robinson.

UMI

other facts under the guise of vital phenomena, but not of denying or excluding them. The trained man, whether investigator, student, naturalist or physicist, never dreams of excluding the facts of the soul, one of which is his own study, and another the pleasure which he derives from it; rather does he bring about an absorption of himself in the world which is at the same time an unconscious absorption of the world in his own personality.

Let us consider that it is to the founder of modern physics that we owe the great discovery[2] of the entirely psychical nature of colors, tastes, odors and sounds, which has opened the way for all the progress of the psychology, of the critique, and even of the metaphysics for which the philosophy of our day considers itself indebted to Berkeley, Hume, Descartes and Kant, and which may still advance so far as to assign resistance and extent[3] to the same class, and to recognize their psychical nature. Indeed the exclusion of psychical facts betrays too arbitrary a character to endure; psychology would return to science by right of observation and classification of facts, because sensation, intellection, pleasure, grief, memory, association and consciousness, are facts no less than those which are recognized as such in the organic and physical realm.

All parties seem to gain by this transaction: positivism loses something of its particularism and becomes truly positive in not excluding any facts; spiritualism returns to science with its unique facts and in a very favorable condition to rid itself of the useless burden of metaphysical presuppositions. After all, this is a good method, to deny nothing, to assume nothing; facts, facts, nothing else but facts. Let us not trouble any more about spirit or soul, or the ego, or the beginning, or the end; all this is beyond the sphere of human knowledge and especially of science.

But I have shown elsewhere that this position can not be sustained and that the conception is inconceivable.[4] Consciousness and the ego are not hypotheses, nor are they the verbal consequences of psychical acts, but rather the condition of those acts. Psychical acts do not exist except in consciousness and they are not conceiv-

[2] I have already stated elsewhere that I would not dare either to affirm or deny but that some philosopher of antiquity had an insight into this truth; perhaps Plato, especially in some part of the Theaetetus.

[3] I tried to do this in a critical analysis of the very admirable essay on *La Matière*, ("Matter") by the late Ernest Neville. This memoir is an extract from the reports of the Academy of Moral and Political Science.

[4] In my address before the Congress of Psychology at Rome, "L'Oggetto Della Psicologia," which has also been translated into French by Professor Beurtier and published in the *Revue de Philosophie*.

able nor can they be verified except in the self and as the acts of a conscious mind. It is not sensation which produces association, but sensation is an association. And it is not association which produces consciousness and the illusion called self; but it is consciousness, it is the self which makes the association. From this follows the irrefutable conclusion, that internal observation is not simply the principal means but the only instrument, the only source of psychology, and that the method of psychological laboratories, in spite of their excellent intentions and of their serious investigations, rests upon a fundamental misunderstanding and an insurmountable contradiction: namely, that the mind and mental facts must be studied in a place where they are not present, and can not be.[5] I based my thesis upon one observation, among others, which now leads me still further, that is to say to the conclusion reached in this paper.

All the other sciences have for their object things which can be observed and which do not suffer any change by being studied. Psychology, on the contrary, changes its own object at every instant and this object is ourselves. Here is a greater difference between psychology and the other sciences, a more fundamental discrepancy, than could be found between any of the other sciences. Geology for instance makes a study of soils, but it does not produce them; psychology not only studies sentiments, affections and moral uplift, but is itself a sentiment; it is itself affection and moral uplift. A glance thrown upon the facts of the soul does not leave these facts the same as they were before: if I perceive that I am ignorant I am no longer so ignorant as I was; if I perceive that I am sinful I would usually commence to improve myself; if I perceive that I have become the toy of passion, that passion then begins to lose its domination; when a people perceives that it is not free it takes one step towards becoming so in recognizing that it is its duty to be free.

Again, the other sciences do not modify us or change us in any particular; while on the other hand we can not study sentiments without arousing them, cultivating them and changing ourselves completely. The objection may be raised that every sort of study, and every additional piece of knowledge changes us and modifies us in so far that we were ignorant before acquiring it and afterwards have become informed to that extent. Thus far one might

[5] This is the subject of my lecture at Heidelberg, *A quoi servent les laboratoires de psychologie?* translated by Lydia G. Robinson and published in *The Monist* of July, 1909, with slight revision, under the title, "Has the Psychological Laboratory Proved Helpful?"

say that all the sciences participate in psychology, rather than to say that psychology is like the other sciences. But the effect is not really the same. If I observe, consider and comprehend a series of numbers, let us say from 7 to 27, at each number I remain almost in the same condition; the change in myself is not proportional to the number of ideas received. On the other hand, if I not only name twenty shades of sentiment, but if I should study them attentively; for instance, love, hate, desire, austerity, joy or bitterness, compassion or hardness of heart, selfishness or devotion, I would myself indeed pass through the twenty shades of sentiment and even more; I myself would become the twenty good things or evil. In the other sciences the personality is forgotten; it is necessary to forget oneself. But in psychology one enters more and more within himself, and this in an active way or in such a way that he knows no alternative, and he does not act at all without being conscious of it. All psychology lies in this, and it is inconceivable in any other way. Hence the fundamental opposition between psychology and the other sciences; hence psychology is something more than a science, it is life, and in this lies its responsibility. The other sciences teach and enlighten us, psychology is ourselves. And just as we are not simple objects of curiosity such as rocks, or the feet of insects, or Niagara Falls, but a will which must perform a duty, and as that duty is fundamentally the uplift of our feelings and of our entire nature; so psychology is not only a simple observation, nor can it be simply the observation of all feeling; but it is careful, scrupulous choice and perfection itself.

People often speak of the applications of psychology; perhaps in doing so they underrate it. Psychology contains its own applications. Every glance which the mind directs upon itself is at the same time both theory and practice. There is no such thing as pedagogy, there is only psychology. We have no certain way to control the feelings of others, even of little children. Those who feel most sure of an infallible result end by accomplishing nothing at all, or else an effect contrary to the one they wish. It is certainly a paradox that the measure of an educator is not what he succeeds in making of his pupils but in what he is himself. An educator is not bound to make others good. The theory and art of education consists in arousing in one's self, and then cultivating and perfecting the kindliest and most generous feelings for children and for everyone, and directing to the service of this good feeling the classification of ideas, the sum total of one's knowledge, and the mastery

of one's self. It is said of some teacher that he has such a way of doing, of managing, of explaining, that he wins others to him, that he helps them, that he makes them understand and makes them love him. Very good, but what is all that if not his own quality? Let us consider the immense weight of an expression very common in all the countries of continental Europe. When they want to exhort a man to be on his good behavior, to give up his vices, and to conquer his passions, they say simply, "Come (or return) to yourself." In employing this strong expression they do not mean the sophistical pride of the man who wishes it understood that he himself is the moral law; but they merely mean to acknowledge that the revelation of moral law as well as every effort to execute it, to correct one's evil inclination or to advance in the path of good, lies simply in an act of reflection, of concentration and of self-examination; that it is not in books, much less in statistical tables, that our law may be found, but within ourselves.

And if this is true of every meditation and every examination of conscience, if it is true of the first steps of consciousness, what a mistake it would be not to recognize that it is still more true of the regular and constant application of the mind to the study of itself, which is called psychology? And that psychology is indeed something different and higher than any other science, for it is the habitual practice of this application together with the transformation and uplift which follows? That it is not only a fact of ethics but its true actualization?

When learning the elements of mental science it is necessary and very useful to first distinguish and clearly establish the existence, the nature, the degrees, and the kinds of sentiments and emotions, and afterwards their moral value. Before the statement is made that such or such a sentiment exists it must be defined, and not until then should it be judged as to whether it is good or bad, whether or not it responds to the ideal of morality, of justice and of perfection. Indeed there is nothing less easy than to confine oneself to this order of procedure, to guard oneself from confusing at first sight the moral consideration, the valuation from the moral point of view, with what ought to be the purely psychological description. But we can not be severe with regard to this confusion and premature synthesis. It is according to the nature of things that moral valuation is not only important from the practical point of view, but it is an integral part of existence and of the nature of psychical facts. Such a sentiment as love for one's neighbor, filial

UMI

affection and love of justice, exists and can be comprehended only in relation to, and dependent upon, its moral value. Psychology, the science of consciousness, should acquire a high consciousness of itself. This would not be only an elegant detail in the department of erudition but an actual ennoblement of the mind and of humanity. When psychology shall have advanced to the true and complete idea of itself it will raise to the loftiest heights all culture and life. Even ethics will gain thereby. Abandoned forever will be the petty etymological peripatetic conception that ethics is the science of customs. As soon as people comprehend that the mind is everything and that everything is in the mind, that the mind is not a means but an end, is not a part of a certain totality, but what has hitherto been conceived as the all is only an effect and a part of mind; we will comprehend that there is no goodness outside of the mind, and we shall easily come to recognize in ethics the science of character and of duty. Every question, every social and political problem should be solved in view of the rights and the finality of the mind. Government, property and customs will be for the mind. The mind has too long been subordinated to its own creations.

L. MICHELANGELO BILLIA.

TURIN, ITALY.

NON-ARISTOTELIAN LOGIC.

Mr. Charles S. S. Peirce adds the following explanation to his comment on non-Aristotelian logic, published on page 45 of the present number of *The Monist*.

"It does not seem to me to have been a lunatic study. On the contrary, perhaps if I had pursued it further, it might have drawn my attention to features of logic that had been overlooked. However, I came to the conclusion that it was not worth my while to pursue that line of thought further. In order to show what sort of false hypotheses they were that I traced out to their consequences, I will mention that one of them was that instead of the form of necessary inference being, as it is, that from A being in a certain relation to B, and B in the same relation to C, it necessarily follows that A is in the same relation to C, I supposed, in one case, that the nature of Reason were such that the fundamental form of inference was, A is in a certain relation to B and B in the same relation to C, whence it necessarily follows that C is in the same relation to A; and I followed out various other similar modifications of logic."

We deny "that from A being in a certain relation to B, and B in the same relation to C, it necessarily follows that A is in the

same relation to C." The statement is not a necessary inference according to the established rules of logic, nor could it be considered Aristotelian.

If A stands in the relation of cousin to B, and B stands in the same relation to C, it does *not* necessarily follow that A is a cousin to C. Or take another instance. If A has a relation to B such as lying 5 feet below B, and B has the same relation to C, it does not follow that A lies 5 feet below C.

We do not venture to discuss the non-Aristotelian substitute, because we are not sure to have grasped the meaning which Mr. Peirce intends to convey.

We will further add that we never used the word "lunatic" in connection with non-Aristotelian logic, nor would we say that it will not be "worth looking into" the theory of a non-Aristotelian logic "notwithstanding its falsity." P. C.

BOOK REVIEWS AND NOTES.

DIE SELEKTIONSTHEORIE; Eine Untersuchung. Von *August Weismann.* Jena: Fischer, 1909. 1 colored plate. Pp. 69.

Professor Weismann's sense of the significance of the theory of selection, as expressed in his contribution to the Darwin centennial memorial prepared by Cambridge University (*Darwin and Modern Science*) is the same as it has been throughout the fifty years of his labors in biological lines, and he himself says that he would never have thought that there could be anything to-day to add on the subject. It was not until he was engaged upon this tribute to Darwin, and even after it was sent to England, that he came to the conclusion that this theory could be established even more firmly than heretofore, that not only can the reality of the selective processes in nature be made to seem in the highest degree probable, as he had hitherto thought, but that they could be actually demonstrated. In his preface Weismann goes on to say: "The logical justification was of course sufficiently sure. When once the hypotheses were granted everything else followed naturally as in a mathematical calculation. But the hypotheses could not always be established. It could not always be demonstrated that the first and smallest beginnings of the countless adaptations which were observed in organisms could be of advantage to their possessor, of such advantage, that is to say, that it could be designated as "worthy of selection"; advantages therefore upon the possession of which depended the life and death of their possessor, at any rate its survival in its posterity. Not until I was thinking over what I had written for the Darwin memorial volume did it occur to me that there are cases in which the value for selection of the stages in the origin and descent of the natural processes of selection lies clearly before our eyes, cases which at the same time decide the question with regard to the reality of such processes....To be sure probability was very strongly on the side of the selection theory even without a formal demonstration. But we know indeed how many considerations and doubts, yea how many of the most highly valued judgments in our own days have repeatedly passed selection by as a "nurse's tale." At any rate certainty is always preferable to even a high degree of probability." The present pamphlet presents Weismann's original essay unchanged, except that it includes the insertion of a few pages which have reference to the new particulars of which he speaks in the above quoted sentences.

M. Théodule Ribot has published a new book on the problems of affective psychology (*Problèmes de psychologie affective,* Paris: Alcan, 1910; price, 2 fr. 50). Some of the essays here collected have been published before, but have been completed by incorporating the results of the author's more recent labors. The first essay discusses the psychology of sentiments in general. The others treat of the emotional aspect of memory, antipathy, pleasure and illusion.

Günther Jacoby, lecturer of philosophy at the University of Greifswald, has written a treatise on pragmatism, (*Der Pragmatismus,* Leipsic, Dürr, 1909) in which he speaks of the new movement very appreciatively. He supports the position of Prof. William James, and endorses his new conception of truth. He looks upon pragmatism as a movement which strikes out on new paths.

UMI